Serendipity Virtual Element Methods

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Assume that: $E = \text{pentagon}$ and we want $\mathbb{P}_2$-accuracy.

We consider 11 degrees of freedom (values at vertexes and midpoints, plus the average on $E$). For every $v \in C^0(\bar{E})$ we define $Dv \in \mathbb{R}^{11}$ to be the nodal values and the average of $v$: $Dv = (Dv)_1, \ldots, (Dv)_{10}, (Dv)_{11}$
Our (local) discrete space will be $\mathbb{R}^{11}$

For $P$ and $Q$ in $\mathbb{R}^{11}$ we want $[P, Q]_E$ to mimic, say

$$\int_E \nabla p \cdot \nabla q \, dE$$

If $p$ is (any) function p.w. $P_2$ on $\partial E$, with $Dp = P$ and $q_2$ a polynomial of degree $\leq 2$ such that $Dq_2 = Q$ then

$$\int_E \nabla p \cdot \nabla q_2 = -\int_E p \Delta q_2 + \int_{\partial E} p \frac{\partial q_2}{\partial n} \sim -P_{11} \int_E \Delta q_2 + \text{Simpson}$$

can be computed without knowing $p$ (but only $P = Dp$).
According to the previous slide, we can compute

\[ [Dp, Dq]_E \sim \int_E \nabla p \cdot \nabla q dE \]

using only the values of \( Dp \) and \( Dq \) in \( \mathbb{R}^{11} \) whenever either \( p \) or \( q \) is a polynomial of degree \( \leq 2 \).

This obviously defines an \( 6 \times 11 \) matrix \( R \), from \( \mathbb{R}^{11} \) to \( \mathbb{R}^6 \) that gives the values of \( [P, D(q_2)]_E \) for \( P \in \mathbb{R}^{11} \) and \( q_2 \in P_2 \sim \mathbb{R}^6 \). Then we can define a sort of scalar product in \( \mathbb{R}^{11} \), for all \( P \) and all \( Q \) in \( \mathbb{R}^{11} \), as

\[ [P, Q]_E := (RP)^T \cdot (RQ) + \text{Stabilization (on the kernel of } R) \]
“Darcy problem”: \( \mathbf{u} = -\nabla p, \text{div}\mathbf{u} = f + \text{B.C.} \)

From now on: \( p = 0 \) on the boundary.

Following B. M. Fraeijs de Veubeke (1965) we observe that: given an approximation \( \lambda_h \) of \( p \) at the inter-element boundaries, and another approximation \( p_h \) of \( p \) inside each element we can deduce an approximation \( \mathbf{u}_h \) of \( \mathbf{u} \) by requiring, in each element \( E \)

\[
\int_E \mathbf{u}_h \cdot \mathbf{v} = \int_E p_h \text{div}\mathbf{v} + \int_{\partial E} \lambda_h (\mathbf{v} \cdot \mathbf{n}) \quad \text{for all } \mathbf{v}.
\]

This defines an approximate gradient

\[
\mathbf{G}_h : (\lambda_h, p_h) \rightarrow \mathbf{u}_h = \mathbf{G}_h(\lambda_h, p_h)
\]
Use of FdV

Darcy problem: \( \mathbf{u} = -\nabla p, \ \text{div} \mathbf{u} = f, \ p = 0 \) at \( \partial \Omega \)

\[
\int_{E} \mathbf{u}_h \cdot \mathbf{v} = \int_{E} p_h \ \text{div} \mathbf{v} + \int_{\partial E} \lambda_h (\mathbf{v} \cdot \mathbf{n}) \text{ for all } \mathbf{v}.
\]

You can then add a discretized conservation equation

\[
\int_{E} \text{div} \mathbf{u}_h \ q = \int_{E} f \ q \text{ for all } q.
\]

Then you must close the system, with as many equations as there are \( \lambda_h \)'s, requiring “continuity” (for \( \mathbf{u}_h \cdot \mathbf{n} \) or for \( p_h \), or for a combination of the two) at the interelement boundaries. You can get zillions of methods.
Approximate Gradient methods

Remember that given a $\lambda_h$ at the inter-element edges (or faces), and a $p_h$ inside each element, we can define an approximate gradient

$$G_h : (\lambda_h, p_h) \rightarrow u_h = G_h(\lambda_h, p_h)$$

by requiring, in each element $E$

$$\int_E G_h \cdot v = \int_E p_h \text{div} v + \int_{\partial E} \lambda_h (v \cdot n) \text{ for all } v$$

where $v$ ranges over the space where you look for $G_h$. Then you look for a pair $(\lambda_h, p_h)$ such that

$$\sum_E \int_E G_h(\lambda_h, p_h) \cdot G_h(\mu_h, q_h) = \int_{\Omega} f \, q_h \text{ for all } (\lambda_h, q_h).$$
Approximate Gradient methods-2

Actually, you also need to add a stabilizing term like

\[ \sum_{E} \int_{E} \mathbf{G}_{h}(\lambda_{h}, p_{h}) \cdot \mathbf{G}_{h}(\mu_{h}, q_{h}) + C h^{-1} \sum_{e} \int_{e} (\lambda_{h} - p_{h})(\mu_{h} - q_{h}) = \int_{\Omega} f \, q_{h}, \quad \forall \mu_{h} \forall q_{h}. \]

Various methods distinguish themselves for the space where you look for the approximate gradient and for the type of stabilization used. You can get zillions of methods
We take again 11 degrees of freedom (values at vertexes and midpoints, + the average on \(E\)). We define the space

\[
V_E := \{ v | \text{such that } v_{|e} \in \mathbb{P}_2(e) \ \forall \text{ edge } e, \text{ and } \Delta v \in \mathbb{P}_0(E) \}
\]

It is easy to see that our 11 d.o.f.s are \(V_E\)-unisolvent.
To every $v \in V_E$ we associate $\Pi_E^2 v \in \mathbb{P}_2(E)$ defined by

$$\int_E \nabla (\Pi_E^2 v - v) \cdot \nabla q_2 = 0 \text{ for all } q_2 \in \mathbb{P}_2(E)$$

$$\int_E (\Pi_E^2 v - v) dE = 0.$$

Note that the quantity ($= \text{right-hand side}$)

$$\int_E \nabla v \cdot \nabla q_2 = -\Delta q_2 \int_E v + \int_{\partial E} v \frac{\partial q_2}{\partial n}$$

is computable (from the dofs) $\forall v \in V_E$ and $\forall q_2 \in \mathbb{P}_2$. 
Basic idea of VEMs-3

Discretized problem: find $p_h \in V$ such that

$$
\sum_E \int_E \nabla \Pi^E p_h \cdot \nabla \Pi^E q_h + S(p_h, q_h) = \sum_E \int_E f \Pi^E q_h \forall q_h \in V
$$

where the stabilizing term $S(p_h, q_h)$ can be taken as

$$
S(p_h, q_h) := \sum_E \left( D(q_h - \Pi^E q_h) \right)^T \cdot \left( D(p_h - \Pi^E p_h) \right)
$$

where $D$ is the degrees of freedom vector defined before.
In summary:

- AD-HOC functions (e.g. Rational, Baricentric, ...); one field and numerical integration
- DG: One field, discontinuous
- MFD: Only dofs, no functions
- HDG WG HHO: Two/three polynomial fields \((\lambda, p, (\mathbf{u}))\)
- VEM: One field, solution of a PDE.
For $k$ integer $\geq 1$ we define

$$V_k(E) = \{ \varphi \in C^0(\bar{E}) : \varphi|_e \in \mathbb{P}_k(e) \forall \text{ edge } e, \ \Delta \varphi \in \mathbb{P}_{k-2}(E) \}.$$ 

The degrees of freedom in $V_k(E)$ are taken as

- the values of $\varphi$ at the vertices,
- $\int_e \varphi q \, ds$ for all $q \in \mathbb{P}_{k-2}(e) \forall \text{ edge } e$,
- $\int_E \varphi q \, dE$ for all $q \in \mathbb{P}_{k-2}(E)$.

It is immediate to verify that the degrees of freedom are unisolvent.
Classical VEM-General case

For $k$ integer $\geq 1$ and $k_\Delta$ integer $\geq -1$ we define

$$V_{k,k_\Delta}(E) = \{ \varphi \in C^0(\bar{E}) : \varphi|_e \in \mathbb{P}_k(e) \forall \text{ edge } e, \Delta \varphi \in \mathbb{P}_{k_\Delta}(E) \}$$

The degrees of freedom in $V_k(E)$ are taken as

- the values of $\varphi$ at the vertices,
- $\int_e \varphi \, q \, ds$ for all $q \in \mathbb{P}_{k-2}(e) \forall \text{ edge } e,$
- $\int_E \varphi \, q \, dE$ for all $q \in \mathbb{P}_{k_\Delta}(E).$

It is immediate to verify that the degrees of freedom are unisolvent. In general it is better to take $k_\Delta \geq k - 2.$
Classical VEM-General case -3D

For $k$ integer $\geq 1$, and $k_\Delta$, $k_f$ integers $\geq -1$ we define

$$V_{k,k_f,k_\Delta}(E) = \{ \varphi \in C^0(\bar{E}) : \varphi|_f \in V_{k,k_f} \forall \text{ face } f, \Delta \varphi \in P_{k_\Delta}(E) \}$$

The following degrees of freedom are unisolvent

- the values of $\varphi$ at the vertices,
- $\int_e \varphi \, q \, ds$ for all $q \in P_{k-2}(e)$ $\forall$ edge $e$,
- $\int_f \varphi \, q \, ds$ for all $q \in P_{k_f}(f)$ $\forall$ face $f$,
- $\int_E \varphi \, q \, dE$ for all $q \in P_{k_\Delta}(E)$. 
VEM and FEM on triangles: degrees of freedom

Figure: Triangles: Classical FEM and Original VEM
VEM and FEM on quads: degrees of freedom

Figure: Quads: Classical FEM and Original VEM
The **reduced** dofs

We choose an $S$ with $(k + 1)(k + 2)/2 \leq S \leq N_E$ ($N_E$ = dimension of $V_{k,k}$), and assume that the degrees of freedom of $V_{k,k}$ are ordered in such a way that the first $S$ dofs $\delta_1, \delta_2, \ldots, \delta_S$ have the following properties:

- **(B)** They include all the *boundary dofs*
- **(S)** $\forall p_k \in \mathbb{P}_k(E)$ we have:
  \[
  \{ \delta_1(p_k) = \delta_2(p_k) = \ldots = \delta_S(p_k) = 0 \} \Rightarrow \{ p_k \equiv 0 \}
  \]

Property (B) is there to ensure *conformity*, and is easy.
The **reduced** dofs - Examples

Once $(\mathcal{B})$ is satisfied, you know that a polynomial $p_k$ that satisfies $\{\delta_1(p_k) = \delta_2(p_k) = \ldots = \delta_S(p_k) = 0\}$ must be identically zero on the boundary. Let’s deal with $(\mathcal{P})$.

To get Property $(\mathcal{P})$, on top of the boundary dofs:

- on a **triangle**, you must include as many internal dofs as the dimension of $\mathbb{P}_{k-3}$,
- on a **square**, you must include as many internal dofs as the dimension of $\mathbb{P}_{k-4}$,
- on a regular **n-gon** you must include as many internal dofs as the dimension of $\mathbb{P}_{k-n}$.

In general (even on very distorted polygons): you must have as many internal dofs as there are $\mathbb{P}_k$-bubbles.
As we are allowing dramatic distortions, the number of $\mathbb{P}_k$-bubbles depends also on the geometry of the element. Here are some examples on quadrilaterals for $k = 4$.

Number of $\mathbb{P}_4$-bubbles on quadrilaterals

1 bubble

3 bubbles

1 bubble

Figure: Allowed distortions for quadrilaterals
The **lazy** choice and the **stingy** choice

As we have seen, the minimum number of internal degrees of freedom that have to be kept depends **on** $k$, and **on the geometry of the element**. Typically: is the dimension of $\mathbb{P}_{k-\eta}$ where $\eta$ is the minimum number of straight lines necessary to cover all the boundary of the element.

In practice, in the **code**, you can either **check every element to compute its** $\eta$ (**stingy choice**), or treat every element as if it was a triangle (**lazy choice**).

One or the other choice could be preferable, depending on the circumstances.
The operator $\mathcal{D}_S$

We assume now that, for a given $k$, we are given a set $\delta_1, \delta_2, \ldots, \delta_S$ of degrees of freedom having the properties $\mathcal{B}$ and $\mathcal{I}$, and we define the operator $\mathcal{D}_S$

$$\mathcal{D}_S : V_{k,k}(E) \to \mathbb{R}^S$$

defined by

$$\mathcal{D}_S \varphi := (\delta_1(\varphi), \ldots, \delta_S(\varphi)).$$

Needless to say, the operator $\mathcal{D}_S$ has the properties

- $\mathcal{D}_S$ can be computed using only the d.o.f $\delta_1, \ldots, \delta_S$,

- $\mathcal{D}_S q = 0 \Rightarrow q = 0$ for all $q \in \mathbb{P}_k$.

$\mathcal{I} \equiv \{ \mathcal{D}_S \text{ is injective when applied to } \mathbb{P}_k \}$
The operator $\mathcal{R}$

We are now going to use $\mathcal{D}_S$ to construct another operator, $\mathcal{R}_S$, as follows: $\forall \varphi \in V_{k,k}(E)$ we define $\mathcal{R}_S \varphi \in \mathcal{P}_k$ by

\[
(\ast) \quad (\mathcal{D}_S \mathcal{R}_S \varphi - \mathcal{D}_S \varphi, \mathcal{D}_S q)_{\mathcal{R}^S} = 0 \quad \forall q \in \mathcal{P}_k,
\]

where $(\cdot, \cdot)_{\mathcal{R}^S}$ is the “Euclidean scalar product” in $\mathcal{R}^S$. Property $\mathcal{I}$ ensures that the matrix

\[
(\mathcal{D}_S p, \mathcal{D}_S q)_{\mathcal{R}^S} \quad p, q \in \mathcal{P}_k
\]

is nonsingular, so that for every $\varphi$ (and hence for every r.h.s. $(\mathcal{D}_S(\varphi), \mathcal{D}_S q)_{\mathcal{R}^S}$) $(\ast)$ has a unique solution $\mathcal{R}_S \varphi$.

$\mathcal{R}_S$ is a projector $V_{k,k} \rightarrow \mathcal{P}_k$
The Serendipity VEM spaces

The operator $\mathcal{R}_S$ that we constructed has the properties:

- $\mathcal{R}_S$ is computable using only the d.o.f. $\delta_1, \ldots, \delta_S$,
- $\mathcal{R}_S q_k = q_k$ for all $q_k \in \mathbb{P}_k$,

that allow us to construct the Serendipity VEM space

$$V^S_k(E) = \{ \varphi \in V_{k,k}(E) \text{ s.t. } \delta_r(\varphi) = \delta_r(\mathcal{R}_S \varphi) \forall r = S+1, \ldots, N_E \}$$

From the first $S$ dofs we can compute $\mathcal{R}_S$ and then using $\mathcal{R}_S$ we compute all the others. And $\mathbb{P}_k \subseteq V^S_k$!
FEM and S-VEM: triangles

FEM k=1

FEM k=2

FEM k=3

VEMS k=1

VEMS k=2

VEMS k=3

Figure: Triangles: Classical FEM and Serendipity VEM
S-FEMS and S-VEM: quads

Figure: Quads: S-FEM (Arnold-Awanou) and S-VEM
Numerical results: Meshes

Figure: Trapezoidal mesh

Figure: Voronoi-Lloyd mesh
Numerical results: $Q_k$-FEM, S-FEM, and S-VEM on quads

Figure: Trapezoidal mesh $k = 3$

Figure: Trapezoidal mesh $k = 4$
Classical VEM and S-VEM (stingy, lazy) on V-Lloyd

Figure: Voronoi-Lloyd mesh $k = 3$

Figure: Voronoi-Lloyd $k = 4$
Serendipity Virtual Elements allow a drastic reduction of the number of internal degrees of freedom.

In 3 dimensions, the degrees of freedom internal to faces are also reduced, making a big difference.

As the older enhanced VEMs, the Serendipity VEMs allow an immediate computation of the $L^2$-projection (of trial and test functions).

On triangles we (finally!) reproduce classical FEMs.

On quads, S-VEM imitate Serendipity FEM, without paying for distortions.

And they preserve all the “geometric freedom” of the original VEMs.