Acknowledgements

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With help from

- Farrah Sadre-Marandi: *ColoState, now MBI (Ohio State)*
- Zhuoran Wang: *ColoState*
Review: Basic concepts of WG finite element methods

Some considerations of WG implementation in C++

Case study: Lowest order WG for Darcy flow
  - on 2-dim meshes with mixed triangles, quadrilaterals thru C++ polymorphism
  - on 3-dim (general) hexahedral meshes

Further considerations of WG implementation and applications
  – e.g., Dimension-independent code?
Model Problem: Elliptic BVP

Boundary value problem (BVP) for 2nd order elliptic equation

\[
\begin{aligned}
\nabla \cdot (-\nabla p) &= f, & \mathbf{x} \in \Omega, \\
p &= 0, & \mathbf{x} \in \Gamma := \partial \Omega.
\end{aligned}
\]  (1)

Variational formulation: Seek \( p \in H^1_0(\Omega) \) such that \( \forall q \in H^1_0(\Omega) \)

\[
\int_{\Omega} \nabla p \cdot \nabla q = \int_{\Omega} f q.
\]

Finite element methods:

- CG: \( V_h \subset H^1_0(\Omega) \), No local conservation or cont. normal flux;
- DG: \( V_h \not\subset H^1_0(\Omega) \), DOFs proliferation, penalty factor;
- WG: Approx. \( \nabla p \) by weak gradient, Many good features!
Weak Functions and Weak Gradient

See Wang, Ye, JCAM (2013)

A weak function on an element $E$ has 2 pieces $\nu = \{\nu^\circ, \nu^\partial\}$

- in interior $\nu^\circ \in L_2(E^\circ)$;
- on element boundary $\nu^\partial \in L_2(\partial E)$.

Note: $\nu^\partial$ may not be the trace of $\nu^\circ$, should a trace be defined.

For any weak function $\nu \in \mathcal{W}(E)$, its weak gradient $\nabla_w \nu$ is defined (interpreted) as a linear functional on $H(\text{div}, E)$:

$$\int_E (\nabla_w \nu) \cdot \mathbf{w} = \int_{\partial E} \nu^\partial (\mathbf{w} \cdot \mathbf{n}) - \int_E \nu^\circ (\nabla \cdot \mathbf{w}) \quad \forall \mathbf{w} \in H(\text{div}, E). \quad (2)$$

Integration By Parts!

Similarly, weak curl/divergence (of vector-valued weak functions)
Let $l, m, n \geq 0$ be integers, $V(E, n)$ a subsp. of $P^n(E)^d (d = 2, 3)$.

- A **discrete weak function** is a weak function $v = \{v^\circ, v^\partial\}$ such that $v^\circ \in P^l(E^\circ), v^\partial \in P^m(\partial E)$.

- For a disc.wk.fxn. $v$, its **discrete weak gradient** is defined by

$$
\int_E \nabla_{w,n} v \cdot w = \int_{\partial E} v^\partial (w \cdot n) - \int_E v^\circ (\nabla \cdot w) \quad \forall w \in V(E, n).
$$

So $\nabla_{w,n} v$ is a lin. comb. of basis fxns. of $V(E, n)$.

**Example:** ($P_0, P_0, RT_0$) on a triangle.

**Implementation** involves

- Three traditional FE spaces: $P^l(E^\circ), P^m(\partial E), V(E, n)$;

- Gram matrix of a basis for $V(E, n)$;

Solving a small SPD lin. sys. (Cholesky factorization).
Let $E = [x_1, x_2] \times [y_1, y_2]$ be a rectangular element.
For WG element $(Q_0, P_0, RT[0])$, there are 5 WG basis functions:
- One constant function in element interior $\phi^\circ$
- One constant function for each of the 4 edges $\phi^\partial_i (i = 1, 2, 3, 4)$.
Their discrete weak gradients are specified as in $RT[0]$.

Figure: 5 basis functions for a WG $(Q_0, P_0, RT[0])$ rectangular element.
Let $E = [x_1, x_2] \times [y_1, y_2]$ be a rectangular element.

$$\nabla_w, n \phi^\circ = 0w_1 + 0w_2 + \frac{-12}{(x_2-x_1)^2}w_3 + \frac{-12}{(y_2-y_1)^2}w_4,$$

$$\nabla_w, n \phi_1^\partial = 0w_1 + \frac{-1}{y_2-y_1}w_2 + 0w_3 + \frac{6}{(y_2-y_1)^2}w_4,$$

$$\nabla_w, n \phi_2^\partial = \frac{1}{x_2-x_1}w_1 + 0w_2 + \frac{6}{(x_2-x_1)^2}w_3 + 0w_4,$$

$$\nabla_w, n \phi_3^\partial = 0w_1 + \frac{1}{y_2-y_1}w_2 + 0w_3 + \frac{6}{(y_2-y_1)^2}w_4,$$

$$\nabla_w, n \phi_4^\partial = \frac{-1}{x_2-x_1}w_1 + 0w_2 + \frac{6}{(x_2-x_1)^2}w_3 + 0w_4.$$
Need for quadrilaterals or more general polygons:
- Flexibility of accommodating problem geometry
- Reducing degrees of freedom

WG elements on quadrilaterals or polygons
- $\text{WG (} P_1, P_1, P_0^2 \text{)},$ See Mu, Wang, Ye, IJNAM (2015)
- $\text{WG (} P_1, P_0, P_0^2 \text{)},$ Shown in Xiu Ye talk yesterday

**Another try:** $\text{WG (} Q_0, P_0, RT_{[0]} \text{)}$ on quadrilaterals?
Figure: Darcy flow in the interstitial space around tumor cells. Source: Rejniak et al., *Frontiers in Oncology*, 2013.
Hexahedral Meshes instead of tetrahedral meshes

For certain complicated domains, e.g.,

Figure: Hossain, Hossainy, Bazilevs, Calo, Hughes, *Comput. Mech.* (2012)
Existing work

- Lin Mu: Matlab code
- Long Chen: *iFEM*
- Liu, Sadre-Marandi: *DarcyLite* (Matlab code)
- ...

**New efforts:**

C++ implementation
Ingredients of FEMs

What are involved?

1. Domain and its boundary;
2. PDEs and boundary conditions
3. Mesh and cells
4. Elements (cells being equipped with basis functions)
5. (Discrete weak) gradient / curl /div of (WG) basis functions
6. Bilinear and linear forms at the element level
   – e.g., grad-grad
7. Assembly at the mesh level;
   Mesh topology info used for global matrix sparsity pattern
   ▶ Element vs its edges
   ▶ Edge vs neighboring edges
   ▶ class SparseBlockMatrix
Multiple Inheritance

- Inheritance from the class for $P^l(E^\circ)$,
  Using the basis functions:
  - their values,
  - Gram matrix for local $L_2$-projection $Q^\circ$, etc.

- Inheritance from the class for $P^m(\partial E)$,
  Using the basis functions:
  - their values,
  - Gram matrix for local $L_2$-projection $Q^\partial$, etc.

- Inheritance from the class for $V(E, n)$,
  Using the basis functions:
  - Gram matrix for solving the small SPD lin. sys. in Eqn.(3),
  - values of basis functions when flux/velocity is needed.

The classes for $P^l(E^\circ), V(E, n)$ are derived classes of the class for geometric cell $E$. 
Classes for Geometric Cells

Just geometric features/properties:
– vertices, volume, outer unit normal on boundary faces, ...

Enumerations of geometric cells in 2-dim, 3-dim:

<table>
<thead>
<tr>
<th>2-dim Cells</th>
<th>3-dim Cells</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tri2d</td>
<td>2-dim triangles</td>
</tr>
<tr>
<td>Rect2d</td>
<td>2-dim rectangles</td>
</tr>
<tr>
<td>Quadri2d</td>
<td>2-dim quadrilaterals</td>
</tr>
<tr>
<td>Polygon</td>
<td>(2-dim) polygons</td>
</tr>
<tr>
<td>Tetra</td>
<td>Tetrahedra</td>
</tr>
<tr>
<td>Brick</td>
<td>3-dim rectangles</td>
</tr>
<tr>
<td>Hexa</td>
<td>Hexahedra (faces maybe not be flat)</td>
</tr>
<tr>
<td>Prism</td>
<td>Cartesian product of a 2-dim cell with an interval</td>
</tr>
</tbody>
</table>

Equipped with shape functions, these cells become finite elements (classes)
WGFEM can be conveniently used on a polytopal mesh with elements of different geometric shapes, e.g., triangles, quadrilaterals, and more general polygons.

Implementation of WG elements can be unified for – triangles, rectangles, quadrilaterals, ... via C++ polymorphism and template (instantiation “on the fly”)

Mesh generation

- PolyMesher: Matlab code (thru Flat File Transfer (FFT))
- TetGen: A tetrahedral mesh generator (FFT, Linking)
- CUBIT/Trelis: A hexahedral mesh generator

Linear Solvers: PETSc

Visualization

- Silo: A mesh & field I/O library, scientific database
- VisIt: An interactive visualization tool (interactive computing!)

FreeFEM++: Use its script language
Case Study: Solving Darcy Equation

By lowest order WG
– No need for stabilization
– Minimum DOFs

Recall: The Darcy flow problem is usually formulated as

\[
\begin{align*}
\nabla \cdot (-K \nabla p) &\equiv \nabla \cdot u = f, \quad x \in \Omega, \\
p & = p_D, \quad x \in \Gamma^D, \\
u \cdot n & = u_N, \quad x \in \Gamma^N,
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^d (d = 2, 3) \) is a bounded polygonal domain, \( p \) the unknown pressure, \( K \) a permeability tensor that is uniformly symmetric positive-definite, \( f \) a source term, \( p_D, u_N \) are respectively Dirichlet and Neumann boundary data, \( n \) the unit outward normal vector on \( \partial \Omega \), which has a nonoverlapping decomposition \( \Gamma^D \cup \Gamma^N \).

Note: Here \( K \) is an order 2 or 3 full SPD matrix.
WGFEM for Darcy: Pressure

**Scheme for Pressure**
Seek $p_h = \{p_h^\circ, p_h^\partial\} \in S_h(l, m)$ such that $p_h^\partial|_{\Gamma^D} = Q_h^\partial p_D$ and

$$A_h(p_h, q) = \mathcal{F}(q), \quad \forall q = \{q^\circ, q^\partial\} \in S_h^0(l, m). \quad (5)$$

where

$$A_h(p_h, q) := \sum_{E \in \mathcal{E}_h} \int_E K \nabla w, n p_h \cdot \nabla w, n q \quad (6)$$

and

$$\mathcal{F}(q) := \sum_{E \in \mathcal{E}_h} \int_E f q^\circ - \sum_{\gamma \in \Gamma_h^N} \int_{\gamma} u_N q. \quad (7)$$
**Velocity**: $L_2$-projection back into subsp. of $V(E, n)$:

$$u_h = Q_h(-K \nabla_w, n p_h),$$

(8)

**Normal flux**: across edges

$$\int_{e \in \partial E} u_h \cdot n_e$$
WG: Bilinear and Linear Forms at the Element Level

**Bilinear form (matrix):** The grad-grad form

\[ \int_E K \nabla w, n p_h \cdot \nabla w, n q \]

**Bilinear form (matrix):** Stabilizer

\[ \sum_{e \in \partial E} \langle Q^p - p^\partial, Q^q - q^\partial \rangle \]

**Linear form (vector):** Source term, Boundary term

\[ \int_E f q^\circ, \int_{\gamma} u_N q^\partial \]

Recall the 3 traditional FE types
Example 1
2-dim, \( \Omega = (0, 1)^2 \),
\[ p(x, y) = \sin(\pi x) \sin(\pi y), \]
Homo. Dirichlet cond. on \( \partial \Omega \)

Mesh
– All triangles
– All rectangles
– All quadrilaterals
– Mix of triangles and quadrilaterals

Perturbation to rect. mesh is controlled by two parameters \( mx, \delta \):

\[
\begin{align*}
\tilde{x} &= \delta \sin(3\pi x) \cos(3\pi y); \\
\tilde{y} &= -\delta \cos(3\pi x) \sin(3\pi y)
\end{align*}
\]
Numer. Ex. 1: Mesh, Pressure, Velocity

\[ n = 16, \text{ Left 1/3rd quadri.}, \text{ Perturbation } \delta = 0.02; \quad \text{Silo, VisIt} \]
1st order convergence in numerical pressure and (edge) normal flux

**Mix of triangles and quadrilaterals:**
Left 1/3rd quadrilaterals
Perturbation parameter $\delta = 0.02$

<table>
<thead>
<tr>
<th>1/h</th>
<th>$| p - p_h^o |_{L_2(\Omega)}$</th>
<th>$\max_{\gamma \in \Gamma_h} | u \cdot n_\gamma - u_h \cdot n_\gamma |_{L_2(\gamma)}$</th>
<th># iterations (CG)</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>3.530e-2</td>
<td>5.754e-1</td>
<td>245</td>
</tr>
<tr>
<td>32</td>
<td>1.766e-2</td>
<td>3.025e-1</td>
<td>503</td>
</tr>
<tr>
<td>64</td>
<td>8.863e-3</td>
<td>1.598e-1</td>
<td>958</td>
</tr>
<tr>
<td>128</td>
<td>4.432e-3</td>
<td>8.057e-2</td>
<td>1916</td>
</tr>
<tr>
<td>rate</td>
<td>1st order</td>
<td>1st order</td>
<td>1st order</td>
</tr>
</tbody>
</table>
1st order convergence in numerical pressure and (edge) normal flux

**All quadrilaterals**: perturbation parameter $\delta = 0.05$

<table>
<thead>
<tr>
<th>1/h</th>
<th>$| p - p^0_h |_{L_2(\Omega)}$</th>
<th>$\max_{\gamma \in \Gamma_h} | u \cdot n_\gamma - u_h \cdot n_\gamma |_{L_2(\gamma)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>4.212e-2</td>
<td>8.656e-1</td>
</tr>
<tr>
<td>32</td>
<td>2.114e-2</td>
<td>4.414e-1</td>
</tr>
<tr>
<td>64</td>
<td>1.058e-2</td>
<td>2.303e-1</td>
</tr>
<tr>
<td>128</td>
<td>5.293e-3</td>
<td>1.169e-1</td>
</tr>
<tr>
<td>rate</td>
<td>1st order</td>
<td>1st order</td>
</tr>
</tbody>
</table>

**Notes:**

  
  *FDA always looks at drug efficacy and toxicity.*

Accuracy of the lowest order WGFEM

Assumptions

$\triangleright \ p \in H^2$

$\triangleright \ \text{Mesh: } \text{Asymptotically parallelogram}$

Proposition. For the elliptic problem and WG scheme with the lowest order elements on mix meshes, there holds 1st order convergence in the numerical pressure and normal flux:

$$\| p - p_h^0 \|_{L^2(\Omega)} = O(h),$$

and

$$\max_{\gamma \in \Gamma_h} \| u \cdot n_\gamma - u_h \cdot n_\gamma \|_{L^2(\gamma)} = O(h).$$
Calling of Libraries

- Tetrahedral meshes generated by TetGen;
- WG FE scheme for numerical pressure;
- A SPD lin. sys. solved by a Conjugate Gradient solver in LinLite, which will be replaced by / cast onto PETSc;
- Numerical pressure plotted by VisIt.

A particular example on \([0, 1]^3\):
- Known exact pressure \(p(x, y, z) = \cos(\pi x) \cos(\pi y) \cos(\pi z)\)
- Simple permeability \(K = \mathbb{I}_3\)
Numer. Ex.2: WG $(P_0, P_0, RT_0)$ Tetra.
Results are as expected

<table>
<thead>
<tr>
<th>$1/h$</th>
<th>$| p - p_h^o |_{L_2(\Omega)}$</th>
<th>$| p - p_h^o |_h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>$4.883e-2$</td>
<td>$4.817e-3$</td>
</tr>
<tr>
<td>16</td>
<td>$2.451e-2$</td>
<td>$1.202e-3$</td>
</tr>
<tr>
<td>32</td>
<td>$1.226e-2$</td>
<td>$2.853e-4$</td>
</tr>
</tbody>
</table>

1st order 2nd order

where the discrete $L_2$ error in pressure is defined as

$$\| p - p_h \|_h^2 = \sum_{E \in \mathcal{E}_h} (p - p_h)^2(E_c)|E|$$

and $E_c$ is the element center.
MFEM: Enhanced $BDDF_1$ elements and special quadrature used

$H(\text{div}), H(\text{curl})$-conforming FE subspace on hexa.
- General trilin. mapping of $RT_{[0]}$ does not contain const. vec.;
- To have const. vec., need a dim-21 subsp. of $RT_{[1]}$ (dim=36).

Nested refinements, Quadratures

Need primary & secondary (e.g., $\hat{x} = \frac{1}{2}$) faces to be flat
Why Hexahedral Meshes

- For complicated domains, e.g., artery (applications in drug delivery).
- Permeabilities in radial, angular, vertical directions are quite different:

\[
K = Q^T K_c Q, \quad K_c = \text{diag}(K_r, K_\theta, K_z), \quad Q = \begin{bmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

- Cylindrical coordinates might better characterize the problems.
- **Choices:**
  - Cartesian FEs on hexahedral meshes
CUBIT/Trelis: Hexahedral Mesh Generator

Mesh
Var: mesh

DB: CubeMinusCylinder.neu

user: jamestliu
Thu Jul 2 01:45:49 2015
A General Hexahedron: Non-flat Face(s)
Hexahedron: Trilinear Mapping from the unit cube $[0, 1]^3$

\[ \mathbf{v} = \mathbf{v}_{000} + \mathbf{v}_a \hat{x} + \mathbf{v}_b \hat{y} + \mathbf{v}_c \hat{z} + \mathbf{v}_d \hat{y} \hat{z} + \mathbf{v}_e \hat{z} \hat{x} + \mathbf{v}_f \hat{x} \hat{y} + \mathbf{v}_g \hat{x} \hat{y} \hat{z}. \]

\[ \mathbf{v}_a = \mathbf{v}_{100} - \mathbf{v}_{000}, \quad \mathbf{v}_d = (\mathbf{v}_{011} - \mathbf{v}_{000}) - (\mathbf{v}_b + \mathbf{v}_c), \]
\[ \mathbf{v}_b = \mathbf{v}_{010} - \mathbf{v}_{000}, \quad \mathbf{v}_e = (\mathbf{v}_{101} - \mathbf{v}_{000}) - (\mathbf{v}_c + \mathbf{v}_a), \]
\[ \mathbf{v}_c = \mathbf{v}_{001} - \mathbf{v}_{000}, \quad \mathbf{v}_f = (\mathbf{v}_{110} - \mathbf{v}_{000}) - (\mathbf{v}_a + \mathbf{v}_b), \]
\[ \mathbf{v}_g = (\mathbf{v}_{111} - \mathbf{v}_{000}) - (((\mathbf{v}_a + \mathbf{v}_b + \mathbf{v}_c) + (\mathbf{v}_d + \mathbf{v}_e + \mathbf{v}_f)). \]

**Fact**: A hexahedron becomes parallelepiped iff $\mathbf{v}_g = \mathbf{0}$.

**Figure**: Naff, Russell, Wilson, *Comput. Geosci.* (2002)
Hexahedron: Jacobian

Recall trilinear mapping $F : [0, 1]^3 \rightarrow E$ (hexahedron)

$\mathbf{v} = \mathbf{v}_{000} + \mathbf{v}_a \hat{x} + \mathbf{v}_b \hat{y} + \mathbf{v}_c \hat{z} + \mathbf{v}_d \hat{y} \hat{z} + \mathbf{v}_e \hat{z} \hat{x} + \mathbf{v}_f \hat{x} \hat{y} + \mathbf{v}_g \hat{x} \hat{y} \hat{z}$.

Tangential vectors

\[
\frac{\partial \mathbf{v}}{\partial \hat{x}} =: \mathbf{X}(\hat{y}, \hat{z}), \quad \frac{\partial \mathbf{v}}{\partial \hat{y}} =: \mathbf{Y}(\hat{z}, \hat{x}), \quad \frac{\partial \mathbf{v}}{\partial \hat{z}} =: \mathbf{Z}(\hat{x}, \hat{y}).
\]

Jacobian matrix $\mathbf{J}_F$

Jacobian determinant $J_F = \left( \mathbf{X}(\hat{y}, \hat{z}) \times \mathbf{Y}(\hat{z}, \hat{x}) \right) \cdot \mathbf{Z}(\hat{x}, \hat{y})$
Piola transformation maps
a vector field \( \hat{\mathbf{w}}(\hat{x}, \hat{y}, \hat{z}) \) on the unit cube \( \hat{E} \) to
a vector field \( \mathbf{w}(x, y, z) \) on a hexahedron \( E \):

\[
\mathbf{w}(x, y, z) := \frac{J_F}{J_{\hat{F}}} \hat{\mathbf{w}}(\hat{x}, \hat{y}, \hat{z})
\]

**Benefits:** Piola transformation preserves the normal flux on each (boundary) face:

\[
\int_{\partial \hat{E}} \hat{\mathbf{w}} \cdot \hat{\mathbf{n}} = \int_{\partial E} \mathbf{w} \cdot \mathbf{n}
\]
Solving for primal variable (pressure)
Approximate $\nabla p$ by discrete weak gradient $\nabla w, n \rho_h$
No use of Piola transformation
Hierarchy of approximations:
from $(Q_0, Q_0, RT_{[0]})$ to $(Q_0, Q_1, P_1^3)$ to higher order
Theorem 1 (Local Mass Conservation)
Let $E$ be any hexahedral element. There holds

$$\int_E f - \int_{\partial E} u_h \cdot n = 0.$$  \hfill (9)

Proof. In Equation (5), take a test function $q$ such that $q|_{E^\circ} = 1$ but vanishes everywhere else. We thus obtain

$$\int_E f = \int_E (K \nabla_w, n p_h) \cdot \nabla_w, n q
= \int_E \mathcal{Q}_h(K \nabla_w, n p_h) \cdot \nabla_w, n q
= - \int_E \nabla \cdot \mathcal{Q}_h(K \nabla_w, n p_h)
= - \int_{\partial E} \mathcal{Q}_h(K \nabla_w, n p_h) \cdot n
= \int_{\partial E} u_h \cdot n.$$
**Theorem 2** (Normal Flux Continuity)

Let $\gamma$ be a (nonplanar) face shared by two hexahedra $E_1, E_2$ and $n_1, n_2$ be respectively the (varying) outward unit normal vector (of $E_1, E_2$. There holds

$$\int_{\gamma} u_h^{(1)} \cdot n_1 + \int_{\gamma} u_h^{(2)} \cdot n_2 = 0. \quad (10)$$

**Proof.** In Equation (5), take a test function $q = (q^\circ, q^\partial)$ such that
- $q^\circ \equiv 0$ on interior of all hexahedra;
- $q^\partial$ nonzero on $\gamma$;
- $q^\partial = 0$ on all faces other than $\gamma$.

Obtain

$$\int_{\gamma} \left( u_h^{(1)} \cdot n_1 + u_h^{(2)} \cdot n_2 \right) q^\partial = 0.$$
Proposition. For \((Q_0, Q_0, RT_0)\):
Let \(p\) be the exact pressure solution of the Darcy BVP and \(p_h\) be WG numerical pressure from Scheme (5) with \((Q_0, Q_0, RT_0)\).
There holds
\[
\|p - p_h\|_{L^2(\Omega)} \leq C h. \tag{11}
\]
Example 3:
\( \Omega = [0, 1]^3 \), \( \mathbf{K} = \mathbb{I}_3 \), known exact pressure

\[ p(x, y, z) = \cos(\pi x) \cos(\pi z) \cos(\pi z). \]

Uniform brick mesh perturbed with \( \delta = 1 \)

\[
\begin{align*}
    x &= \hat{x} + \delta \times 0.03 \sin(3\pi \hat{x}) \cos(3\pi \hat{y}) \cos(3\pi \hat{z}) \\
    y &= \hat{y} - \delta \times 0.04 \cos(3\pi \hat{x}) \sin(3\pi \hat{y}) \cos(3\pi \hat{z}) \\
    z &= \hat{z} + \delta \times 0.05 \cos(3\pi \hat{x}) \cos(3\pi \hat{y}) \sin(3\pi \hat{z})
\end{align*}
\]

See also Wheeler, Xue, Yotov, *Numer. Math.* (2012)
Numer. Ex.3: WG Hexa. \((Q_0, Q_0, RT_{[0]})\) Numer. Pres.
Numer. Ex. 3: WG Hexa. \( (Q_0, Q_0, RT_{[0]}) \): Error

Table: Convergence rates of errors

<table>
<thead>
<tr>
<th>1/h</th>
<th>( | p - p_h |_{L^2(\Omega)} )</th>
<th>( | p - p_h |_h )</th>
<th>Flux discrepancy</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>7.001e-2</td>
<td>8.042e-3</td>
<td>1.748e-15</td>
</tr>
<tr>
<td>16</td>
<td>3.562e-2</td>
<td>2.108e-3</td>
<td>2.367e-15</td>
</tr>
<tr>
<td>32</td>
<td>1.790e-2</td>
<td>5.358e-4</td>
<td>3.939e-15</td>
</tr>
<tr>
<td>64</td>
<td>8.965e-3</td>
<td>1.345e-4</td>
<td>3.380e-15</td>
</tr>
</tbody>
</table>

1st order 2nd order Double precision (zero)

where the discrete \( L_2 \) error in pressure is defined as

\[
\| p - p_h \|_h^2 = \sum_{E \in \mathcal{E}_h} (p - p_h)^2(E_c) |E|
\]

and \( E_c \) is the element center.
Questions

(i) Dimension-independent implementation?
   Unified treatment of 2-dim & 3-dim elements
   (like deal.II)

(ii) Unified treatment of simplicial and tensor-type elements?
    Simplicial: triangle, tetrahedron
    Tensor-type: rectangle (2d), brick (3d)
    More general: quadrilateral, hexahedral, ...

(iii) Unified treatment of scalar/vector-valued WG elements?
    Weak gradient/div/curl

> ...
Further Applications of WG FEMs

- WG for time-dependent conviction-diffusion (transport) eqn.
  Space-time WG finite elements?
  Spatial WG finite elements + characteristic tracking?
- WG applied to drug transport problems
  - Darcy equation
  - Stokes equation
  - Transport equation
  - Two-phase problems
  - WG (Darcy eqn.) + FVM (transport)
- ?
WG + chara. tracking

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