A review of Hybrid High-Order methods: formulations, computational aspects, links with other methods

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Outline

Literature review

Setting

The HHO method in primal form

Links HHO/other polytopal discretization methods

The HHO method in mixed form

Conclusion
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Lowest-order polytopal discretization methods

Finite Volume methods

- Mixed/Hybrid Finite Volume (M/HFV) [Droniou and Eymard, 06 + Eymard, Gallouët, and Herbin, 10]

Mimetic/Compatible methods

- Mimetic Finite Difference (MFD) [Brezzi, Lipnikov, and Shashkov, 05 + Beirão da Veiga, Lipnikov, and Manzini, 14]
  \(\leadsto\) equivalence with M/HFV [Droniou, Eymard, Gallouët, and Herbin, 10]
- Discrete Geometric Approach (DGA) [Codecasa, Specogna, and Trevisan, 10]
- Compatible Discrete Operator (CDO) [Bonelle and Ern, 14]

Non-conforming/penalized methods

- Cell-Centered Galerkin (CCG) [Di Pietro, 12]

Unifying frameworks

- Gradient Schemes [Droniou, Eymard, Gallouët, and Herbin, 13]
- CDO
High-order polytopal discretization methods

Finite Element (FE) methods [Wachspress, 75 + Tabarraei and Sukumar, 04 + Gillette, Rand, and Bajaj]

Virtual Element (VE) methods
- Conf. VE [Beirão da Veiga, Brezzi, Cangiani, Manzini, Marini, and Russo, 13]
- Non-conf. VE [Ayuso de Dios, Lipnikov, and Manzini]
- Unified framework [Cangiani, Manzini, and Sutton]


Hybridizable DG (HDG) methods [Cockburn, Gopalakrishnan, and Lazarov, 09]

Weak Galerkin (WG) methods [Wang and Ye, 13]

Hybrid High-Order (HHO) methods [Di Pietro, Ern, and Lemaire, 14]
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Model problem

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be an open, connected, bounded polytopal domain.

**Problem:** Find a potential $u : \Omega \to \mathbb{R}$ such that

$$\begin{cases}
- \text{div}(\mathbb{M} \nabla u) = f & \text{in } \Omega \\
 u = 0 & \text{on } \partial \Omega
\end{cases}$$

$s.t.$ $f \in L^2(\Omega)$, $\mathbb{M}$ symmetric, piecewise Lipschitz, matrix-valued coeff. s.t. for a.e. $x \in \Omega$, and all $\xi \in \mathbb{R}^d$ s.t. $|\xi| = 1$,

$$0 < \mu_b \leq \mathbb{M}(x)\xi : \xi \leq \mu_\# < +\infty$$
Admissible mesh sequences

Definition
The mesh sequence \((\mathcal{T}_h)_{h \in \mathcal{H}}\) is admissible if, for all \(h \in \mathcal{H}\), \(\mathcal{T}_h\) is a finite collection of polygons/polyhedra \(T\) s.t. \(\overline{\Omega} = \bigcup_{T \in \mathcal{T}_h} \overline{T}\), and \(\mathcal{T}_h\) admits a matching simplicial submesh \(\mathcal{S}_h\) such that \((\mathcal{S}_h)_{h \in \mathcal{H}}\) is

- **shape-regular** in the usual sense of Ciarlet;
- **contact-regular**: every simplex \(S \subseteq T\) is s.t. \(h_S \approx h_T\).

\[\leadsto \text{Assumption: } M \in \left[\mathbb{P}_d^0(\mathcal{T}_h)\right]_{\text{sym}}^{d \times d} \quad \forall h \in \mathcal{H}, \text{ and } \forall T \in \mathcal{T}_h, \ M_T := M|_T \text{ is s.t.} \]

\[\mu_{b,T} \leq M_T \xi \cdot \xi \leq \mu_{\#_T} \text{ (local anisotropy ratio: } \rho_T := \mu_{\#_T} / \mu_{b,T})\]

**Figure**: Admissible meshes in 2D
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Discrete unknowns \((k \geq 0)\)

Figure: DoFs associated with potential unknowns, \(d = 2\)

Local hybrid set of potential unknowns

\[
U_k^T := \mathcal{P}_d^k(T) \times \bigg\{ \prod_{F \in \mathcal{F}_T} \mathcal{P}_{d-1}^k(F) \bigg\}
\]

Local reduction operator

\[
\mathbb{I}_T^k : H^1(T) \rightarrow U_k^T \text{ s.t., for all } v \in H^1(T),
\quad \mathbb{I}_T^k v := \left( \Pi_T^k v, \left( \Pi_{F}^k v \right)_{F \in \mathcal{F}_T} \right)
\]
Potential reconstruction operator

Local potential reconstruction operator: \( p_{T}^{k+1} : U_{T}^{k} \rightarrow \mathbb{P}_{d}^{k+1}(T) \)

For \( v_{T} = (v_{T}, v_{F_{T}}) \in U_{T}^{k} \), \( p_{T}^{k+1}v_{T} \in \mathbb{P}_{d}^{k+1}(T) \) is s.t. \( \int_{T} p_{T}^{k+1}v_{T} = \int_{T} v_{T} \) and satisfies, for all \( w \in \mathbb{P}_{d}^{k+1}(T) \),

\[
\left( \mathbb{M}_{T} \nabla p_{T}^{k+1}v_{T}, \nabla w \right)_{T} = -\left( v_{T}, \text{div(} \mathbb{M}_{T} \nabla w \text{)} \right)_{T} + \sum_{F \in F_{T}} \left( v_{F}, \mathbb{M}_{T} \nabla w \cdot n_{T,F} \right)_{F}
\]

\( \rightarrow \) diffusivity included in reconstruction operator

Computation
Requires to invert a SPD matrix of size \( N_{(k+1),d} \) with \( N_{k,l} := \text{dim}(\mathbb{P}^{k}_{l}) \)

Approximation
For all \( v \in H^{k+2}(T) \), the following holds:

\[
\| v - p_{T}^{k+1}I_{T}^{k}v \|_{T} + h_{T}\| \nabla (v - p_{T}^{k+1}I_{T}^{k}v) \|_{T} \lesssim \rho_{T}^{1/2}h_{T}^{k+2}\| v \|_{k+2,T}
\]
\[ a_T(u_T, v_T) := (M_T \nabla p_T^{k+1} u_T, \nabla p_T^{k+1} v_T)_T + j_T(u_T, v_T) \]

Local stabilization bilinear form: \( j_T : U_T^k \times U_T^k \to \mathbb{R} \)

For all \( u_T, v_T \in U_T^k \),

\[ j_T(u_T, v_T) := \sum_{F \in F_T} \frac{\mu_{T,F}}{h_F} (\Pi_F^k (q_T^{k+1} u_T - u_F), \Pi_F^k (q_T^{k+1} v_T - v_F))_F, \]

where \( \mu_{T,F} := M_T n_F \cdot n_F \), and \( q_T^{k+1} w_T := w_T + (p_T^{k+1} w_T - \Pi_T^k p_T^{k+1} w_T) \)

\( \leadsto \) the use of \( \Pi_F^k \) is reminiscent of Lehrenfeld-Schöberl stabilization for HDG [Lehrenfeld, 10]

\( \leadsto \) the operator \( q_T^{k+1} \) is new and opens the door to lower-order cell unknowns

**Approximation**

For all \( v \in H^{k+2}(T) \), the following bound holds:

\[ j_T(\underline{1}_T^k v, \underline{1}_T^k v)^{1/2} \lesssim \mu_{u,T}^{1/2} \rho_T^{1/2} h_T^{k+1} \| v \|_{k+2,T} \]
Global hybrid set of potential unknowns

\[ \underline{U}_h^k := \mathbb{P}_d^k(\mathcal{T}_h) \times \mathbb{P}_{d-1}^k(\mathcal{F}_h) \]

Discrete problem

Find \( u_h \in \underline{U}_h^k, 0 \) s.t.

\[ a_h(u_h, v_h) = (f, v_{\mathcal{T}_h}) \quad \text{for all} \quad v_h \in \underline{U}_h^k, 0 \]

with \( a_h(u_h, v_h) := \sum_{T \in \mathcal{T}_h} a_T(u_T, v_T) \)

Stability

\[ \rho_T^{-1} \| \mathbb{M}_T^{1/2} \nabla v_T \|_{T}^2 + \rho_T^{-1} \sum_{F \in \mathcal{F}_T} \frac{\mu_{T,F}}{h_F} \| v_T - v_F \|_{F}^2 \lesssim a_T(v_T, v_T) \]
Error estimates

Theorem (Energy-norm error estimate)

Assume $u \in U_0 \cap H^{k+2}(\mathcal{T}_h)$. Then,

$$
\| M^{1/2}(\nabla u - \nabla h_{P(T)}^{k+1} u_h) \| \lesssim \left\{ \sum_{T \in \mathcal{T}_h} \mu_{\#}, T \rho_T^2 h_T T^{2(k+1)} \| u \|^2_{k+2,T} \right\}^{1/2}
$$

Theorem ($L^2$-norm error estimate)

Assume elliptic regularity under the form $\| z(g) \|_{2,T_h} \lesssim \mu_b^{-1}\| g \|$. Assume $f \in H^{k+\delta}(\Omega)$, with $\delta = 0$ for $k \geq 1$ and $\delta = 1$ for $k = 0$. Then,

$$
\mu_b \| \Pi_{T_h}^k u - u_{T_h} \| \lesssim \mu_{\#}^{1/2} \rho h \left\{ \sum_{T \in \mathcal{T}_h} \mu_{\#}, T \rho_T^2 h_T T^{2(k+1)} \| u \|^2_{k+2,T} \right\}^{1/2} + \| f \|_{k+\delta}
$$
Local conservativity

1 - Introduce the local bilinear form
\[ \hat{a}_T(w_T, v_T) := (M_T \nabla p_T^{k+1} w_T, \nabla p_T^{k+1} v_T)_T + \sum_{F \in \mathcal{F}_T} \frac{\mu_{T,F}}{h_F} (w_T - w_F, v_T - v_F)_F \]

2 - Define the local isomorphism \( c_T^k : U_T^k \to U_T^k \) s.t.
\[ \hat{a}_T(c_T^k w_T, v_T) = \hat{a}_T(w_T, v_T) + j_T(w_T, v_T) \quad \forall \ v_T \in U_T^k \]

3 - Define the local gradient recons. operator \( G_T^{k+1} := \nabla (p_T^{k+1} \circ c_T^k) \)

Lemma
For all \( T \in \mathcal{T}_h \), the following local equilibrium holds:
\[ (M_T G_T^{k+1} u_T, \nabla v_T)_T - \sum_{F \in \mathcal{F}_T} (\Phi_{T,F}(u_T), v_T)_F = (f, v_T)_T \quad \forall \ v_T \in P_d^k(T) \]

with conservative numerical flux
\[ \Phi_{T,F}(u_T) := M_T G_T^{k+1} u_T \cdot n_T,F - \frac{\mu_{T,F}}{h_F} [(c_T^k u_T - u_T) - (c_F^k u_T - u_F)] \]
Solution strategy

Offline step $\leadsto$ 2 fully parallelizable and $f$-independent substeps

- 1 - Compute the potential reconstruction operator $p_{T_h}^{k+1}$
  $\leadsto$ invert $\text{card}(T_h)$ SPD matrices of size $N_{(k+1),d}$

- 2 - For all $T \in T_h$, compute the trace-based $t_T^k : \mathbb{P}_{d-1}^k(F_T) \to \mathbb{P}_d^k(T)$ and datum-based $d_T^k : \mathbb{P}_d^k(T) \to \mathbb{P}_d^k(T)$ lifting operators s.t.
  \[
  t_T^k w_{F_T} \in \mathbb{P}_d^k(T) \text{ solves } \quad a_T((t_T^k w_{F_T}, 0), (v_T, 0)) = -a_T((0, w_{F_T}), (v_T, 0)) \quad \forall v_T \in \mathbb{P}_d^k(T)
  \]
  \[
  d_T^k \Psi_T \in \mathbb{P}_d^k(T) \text{ solves } \quad a_T((d_T^k \Psi_T, 0), (v_T, 0)) = (\Psi_T, v_T)_T \quad \forall v_T \in \mathbb{P}_d^k(T)
  \]
  $\leadsto$ invert $\text{card}(T_h)$ SPD matrices of size $N_{k,d}$

Online step

- 1 - Given $f \in L^2(\Omega)$, compute its $L^2$-orthogonal projection $\Pi_{T_h}^k f$ onto $\mathbb{P}_d^k(T_h)$

- 2 - Solve the global problem: Find $u_{F_h} \in \mathbb{P}_{d-1,0}^k(F_h)$ s.t.
  \[
  a_h(t_{T_h}^k u_{F_h}, t_{T_h}^k v_{F_h}) = (\Pi_{T_h}^k f, t_{T_h}^k v_{F_h}) \quad \forall v_{F_h} \in \mathbb{P}_{d-1,0}^k(F_h)
  \]
  where $t_{T_h}^k w_{F_h} := (t_{T_h}^k w_{F_h}, w_{F_h})$
  $\leadsto$ solve a linear system of size $\approx \text{card}(F_h) \times N_{k,(d-1)}$

- 3 - Compute the discrete solution according to $\underline{u}_h = (t_{T_h}^k u_{F_h} + d_{T_h}^k \Pi_{T_h}^k f, u_{F_h})$
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Assume here $\mathbb{M} = \text{Id}_d$.

The HHO(l) family: $U_{T}^{k,l} := \mathbb{P}_{d}(T) \times \mathbb{P}_{d-1}^{k}(\mathcal{F}_{T})$, $l \in \{k - 1, k, k + 1\}$

- The choice $l = k$ corresponds to the original HHO method
- The Non-conf. VE method is, up to equivalent stabilization, a member of the HHO family (for $l = k - 1$) [Cockburn, Di Pietro, and Ern, 15]
- The final system has the same size for any choice of $l$

Similarities/differences among discontinuous skeletal methods

- HHO fits into the HDG framework [Cockburn, Di Pietro, and Ern, 15]
- The flux unknowns associated with HDG and WG belong to $\mathbb{P}_{d}^{k}(T)^{d}$, whereas the flux unknowns associated with HHO belong to $\nabla \mathbb{P}_{d}^{k+1}(T)$: smaller local problems must be solved to eliminate flux unknowns in HHO
Outline

Literature review

Setting

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Discrete unknowns \((k \geq 0)\)

Local \{hybrid set of flux unknowns\} + \{set of potential unknowns \(:= \mathbb{P}^k_d(T)\)\}

\[
\mathbb{S}^k_T := \mathbb{M}_T \nabla \mathbb{P}^k_d(T) \times \left\{ \prod_{F \in \mathcal{F}_T}^k \mathbb{P}^k_{d-1}(F) \right\}
\]

Local reduction operator: \(S^+(T) := \{ t \in L^q(T) \mid \text{div} \ t \in L^2(T) \}, \ q > 2 \)

\[
\mathbb{I}^k_T : S^+(T) \rightarrow \mathbb{S}^k_T \quad \text{s.t.,} \quad \forall t \in S^+(T), \quad \mathbb{I}^k_T t := \left( \mathbb{M}_T \nabla y, \left( \prod^k_F \mathbb{P}^k_d(t \cdot n_F) \right)_{F \in \mathcal{F}_T} \right),
\]

with \(y \in \mathbb{P}^k_d(T)\) a solution of \((\mathbb{M}_T \nabla y, \nabla w)_T = (t, \nabla w)_T \quad \forall w \in \mathbb{P}^k_d(T)\)
Divergence reconstruction operator

Local divergence reconstruction operator: $D^k_T : S^k_T \rightarrow P^k_d(T)$

For all $t_T = (t_T, t_{\mathcal{T}_T}) \in S^k_T$, $D^k_T t_T \in P^k_d(T)$ satisfies, for all $w \in P^k_d(T)$,

$$
(D^k_T t_T, w)_T = -(t_T, \nabla w)_T + \sum_{F \in \mathcal{T}_T} (t_F \varepsilon_{T,F}, w)_F
$$

where $\varepsilon_{T,F} := n_F \cdot n_{T,F}$

Computation

Requires to invert a SPD matrix of size $N_{k,d} = \binom{k+d}{k}$

Commuting property

The following holds for all $t \in S^+(T)$:

$$
D^k_T \Pi^k_T t = \Pi^k_T (\text{div} \ t)
$$

$\Rightarrow$ this ensures inf-sup stability for the discretization
Flux reconstruction operator

Local flux reconstruction operator: \( F_T^{k+1} : S_T^k \to M_T \nabla P_d^{k+1}(T) \)

For all \( t_T = (t_T, t_F) \in S_T^k \), \( F_T^{k+1} t_T := M_T \nabla z \), where \( z \in P_d^{k+1}(T) \) satisfies, for all \( w \in P_d^{k+1}(T) \),

\[
(M_T \nabla z, \nabla w)_T = -(D_T^k t_T, w)_T + \sum_{F \in F_T} (t_F \varepsilon_T, F, w)_F
\]

\( \sim \) diffusivity included in reconstruction operator

Computation

Requires to invert a SPD matrix of size \( N_{(k+1),d} \)

Approximation

For all \( v \in H^{k+2}(T) \), letting \( t := M_T \nabla v \), the following holds for all \( F \in F_T \):

\[
\| M_T^{-1/2} (t - F_T^{k+1} 1_T t) \|_T + h_F^{1/2} \mu_{T,F}^{-1/2} \| (t - F_T^{k+1} 1_T t) \cdot n_F \|_F \lesssim \rho_T^{1/2} \mu^{1/2}_{\#T} h_T^{k+1} \| v \|_{k+2,T}
\]
Stabilization

\[ H_T(s_T, t_T) := (\mathbb{M}_T^{-1} F_T^{k+1} s_T, F_T^{k+1} t_T)_T + J_T(s_T, t_T) \]

Local stabilization bilinear form: \( J_T : S^k_T \times S^k_T \rightarrow \mathbb{R} \)

For all \( s_T, t_T \in S^k_T \),

\[ J_T(s_T, t_T) := \sum_{F \in \mathcal{F}_T} \frac{h_F}{\mu_{T,F}} \left( (F_T^{k+1} s_T) \cdot n_T - s_T, (F_T^{k+1} t_T) \cdot n_T - t_T \right)_F \]

Approximation

For all \( v \in H^{k+2}(T) \), the following bound holds with \( t := \mathbb{M}_T \nabla v \):

\[ J_T(\mathbb{I}_T^k t, \mathbb{I}_T^k t)^{1/2} \leq \rho_T^{1/2} \mu_{\#T}^{1/2} h_T^{k+1} \| v \|_{k+2, T} \]
Discrete problem

**Mixed weak formulation of (1)**

Let $S := H(\text{div}, \Omega)$, $V := L^2(\Omega)$. Find $(s, u) \in S \times V$ s.t.

$$
\begin{cases}
(M^{-1} s, t) + (u, \text{div } t) = 0 & \forall t \in S \\
-(\text{div } s, v) = (f, v) & \forall v \in V
\end{cases}
$$

---

**Global** \{hybrid set of flux unknowns\} \+ \{set of potential unknowns\} := $P^k_d(\mathcal{T}_h)$

$$
S^k_{sh} := M \nabla P^k_d(\mathcal{T}_h) \times P^k_d(\mathcal{F}_h)
$$

**Discrete problem:** Find $(s_h, u_{\mathcal{T}_h}) \in S^k_{sh} \times P^k_d(\mathcal{T}_h)$ s.t.

$$
\begin{cases}
H_h(s_h, t_{\mathcal{T}_h}) + (u_{\mathcal{T}_h}, D^k_{\mathcal{T}_h} t_{\mathcal{T}_h}) = 0 & \forall t_{\mathcal{T}_h} \in S^k_{sh} \\
-(D^k_{\mathcal{T}_h} s_h, v_{\mathcal{T}_h}) = (f, v_{\mathcal{T}_h}) & \forall v_{\mathcal{T}_h} \in P^k_d(\mathcal{T}_h)
\end{cases}
$$

with $H_h(s_h, t_{\mathcal{T}_h}) := \sum_{T \in \mathcal{T}_h} H_T(s_T, t_T)$

**Stability**

$$
\mu^{-1}_{\#, T} \| t_{\mathcal{T}_h} \|_T^2 + \mu^{-1}_{\#, T} \sum_{F \in \mathcal{F}_T} h_F \| t_F \|_F^2 \lesssim H_T(t_{\mathcal{T}_h}, t_{\mathcal{T}_h}) + \text{inf-sup}
$$
Error estimates

**Theorem (Error estimate for the flux)**

Assume \( u \in V \cap H^{k+2}(\mathcal{T}_h) \) and \( s \in S \cap S^+(\mathcal{T}_h) \). Then,

\[
\| M^{-1/2} (s - F_{\mathcal{T}_h}^{k+1} s_{h}) \| \lesssim \left\{ \sum_{T \in \mathcal{T}_h} \mu_{\#,T} \rho_T h_T^{2(k+1)} \| u \|_{k+2,T}^2 \right\}^{1/2}
\]

**Theorem (Supercloseness of the potential)**

Assume elliptic regularity under the form \( \| z(g) \|_{2,\mathcal{T}_h} \lesssim \mu_b^{-1} \| g \| \). Assume \( f \in H^{k+\delta}(\Omega) \), with \( \delta = 0 \) for \( k \geq 1 \) and \( \delta = 1 \) for \( k = 0 \). Then,

\[
\mu_b \| \Pi_{\mathcal{T}_h}^k u - u_{\mathcal{T}_h} \| \lesssim \mu_{\#,\rho}^{1/2} \rho^{1/2} h \left\{ \sum_{T \in \mathcal{T}_h} \mu_{\#,T} \rho_T h_T^{2(k+1)} \| u \|_{k+2,T}^2 \right\}^{1/2} + h^{k+2} \| f \|_{k+\delta}
\]
Characterization of the solution

Unpatched global hybrid set of flux unknowns

\[
\tilde{S}_h^k := \bigotimes_{T \in \mathcal{T}_h} S_T^k, \quad \tilde{Z}_h^k := \left\{ \tilde{t}_h \in \tilde{S}_h^k \mid \sum_{T \in \mathcal{T}_F} \tilde{t}_{T,F} = 0, \forall F \in \mathcal{F}_h^i \right\}
\]

\[\leadsto\text{natural isomorphism } L_h^k \text{ from } \tilde{Z}_h^k \text{ to } S_h^k\]

Potential-to-flux mapping operator: \( \tilde{\varsigma}_h^k : U_h^k \rightarrow \tilde{S}_h^k\)

For \( v_T \in U_T^k \), \( \tilde{\varsigma}_T^k v_T \) satisfies, for all \( \tilde{t}_T \in S_T^k \),

\[
H_T(\tilde{\varsigma}_T^k v_T, \tilde{t}_T) = -(v_T, D_T^k \tilde{t}_T)_T + \sum_{F \in \mathcal{F}_T} (\tilde{t}_{T,F}, v_F)_F
\]

\[\leadsto \text{There holds: for all } v_T \in U_T^k, \ (F_T^{k+1} \circ \tilde{\varsigma}_T^k) v_T = M_T \nabla p_T^{k+1} v_T
\]

Characterization of the solution

Let \( \tilde{u}_h \in U_{h,0}^k \) solve \( A_h(\tilde{u}_h, v_h) = (f, v_{\mathcal{T}_h}) \) for all \( v_h \in U_{h,0}^k \), with

\[
A_h(\tilde{u}_h, v_h) := \sum_{T \in \mathcal{T}_h} (M_T \nabla p_T^{k+1} \tilde{u}_T, \nabla p_T^{k+1} v_T)_T + \sum_{T \in \mathcal{T}_h} J_T(\tilde{\varsigma}_T^k \tilde{u}_T, \tilde{\varsigma}_T^k v_T)
\]

Then, there holds \( \tilde{\varsigma}_h^k \tilde{u}_h \in \tilde{Z}_h^k \) and \((s_h, u_{\mathcal{T}_h}) = (L_h^k(\tilde{\varsigma}_h^k \tilde{u}_h), \tilde{u}_{\mathcal{T}_h})\).
Solution strategy

Offline step $\leadsto$ 4 fully parallelizable and $f$-independent substeps

- 1 - Compute the divergence reconstruction operator $D^k_{T_h}$
  $\leadsto$ invert $\text{card}(T_h)$ SPD matrices of size $N_{k,d}$
- 2 - Compute the flux reconstruction operator $F^{k+1}_{T_h}$
  $\leadsto$ invert $\text{card}(T_h)$ SPD matrices of size $N_{(k+1),d}$
- 3 - Compute the potential-to-flux mapping operator $\tilde{\varsigma}^k_{T_h}$
  $\leadsto$ invert $\text{card}(T_h)$ SPD matrices of size $N_{k,d} + \text{card}(\mathcal{F}_T)N_{k,(d-1)}$
- 4 - For all $T \in T_h$, compute the lifting operators $t^k_T : \mathcal{P}_{d-1}^k(\mathcal{F}_T) \to \mathcal{P}_d^k(T)$ and $d^k_T : \mathcal{P}_d^k(T) \to \mathcal{P}_d^k(T)$ associated with the bilinear form $A_T$
  $\leadsto$ invert $\text{card}(T_h)$ SPD matrices of size $N_{k,d}$

Online step

- 1 - Given $f \in L^2(\Omega)$, compute its $L^2$-orthogonal projection $\Pi^k_{T_h} f$ onto $\mathcal{P}_d^k(T_h)$
- 2 - Solve the global coercive problem: Find $\tilde{u}_{\mathcal{F}_h} \in \mathcal{P}_{d-1,0}^k(\mathcal{F}_h)$ s.t.
  $$A_h(t^k_{T_h} \tilde{u}_{\mathcal{F}_h}, t^k_{T_h} \nu_{\mathcal{F}_h}) = (\Pi^k_{T_h} f, t^k_{T_h} \nu_{\mathcal{F}_h}) \quad \forall \nu_{\mathcal{F}_h} \in \mathcal{P}_{d-1,0}^k(\mathcal{F}_h)$$
  $\leadsto$ solve a linear system of size $\approx \text{card}(\mathcal{F}_h) \times N_{k,(d-1)}$
- 3 - Compute the discrete solution according to $(s_h, u_{T_h}) = (L^k_{T_h} (\tilde{\varsigma}^k_{T_h} \tilde{u}_h), \tilde{u}_{T_h})$, with $\tilde{u}_h = (t^k_{T_h} \tilde{u}_{\mathcal{F}_h} + d^k_{T_h} \Pi^k_{T_h} f, \tilde{u}_{\mathcal{F}_h})$
Outline

Literature review

Setting

The HHO method in primal form

Links HHO/other polytopal discretization methods

The HHO method in mixed form

Conclusion
Assets of HHO methods

- Capable of handling general polytopal meshes
- Dimension-independent construction
- Arbitrary approximation order (starting from $k = 0$)
- Physical fidelity
  - Local conservation
  - Robustness w.r.t. physical parameters in various situations: heterogeneous/anisotropic diffusion, quasi-incompressible linear elasticity, advection-dominated transport, Stokes flow driven by large irrotational forces, Biot’s model of poroelasticity (coupled with DG)...
- Reduced computational cost after static condensation

$$N_{\text{DoFs}}^{\text{HHO}} \approx \frac{1}{2} k^2 \text{card}(\mathcal{F}_h) \quad \text{vs.} \quad N_{\text{DoFs}}^{\text{DG}} \approx \frac{1}{6} k^3 \text{card}(\mathcal{T}_h)$$

- Natural offline/online solution strategy: adapted to the multi-query context
THANK YOU FOR YOUR ATTENTION

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