Polygonal Splines and Their Applications for Numerical Solution of PDE

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Multivariate splines, usually defined on a triangulation in 2D, or a tetrahedral partition in 3D, or spherical surface, or a simplicial partition in $\mathbb{R}^n$, have been developed for 30 years and they are extremely useful to various numerical applications such as computer aided geometric design, numerical solutions of various linear and nonlinear partial differential equations, scattered data interpolation and fitting, image enhancements, spatial statistical analysis and data forecasting, and etc..
Triangulated Splines for Applications

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Spline Method for PDE

GBC

BB functions

Polygonal Spline Space

Hexahedral Spline Space

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image data

2D spline interpolation

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Triangulated Splines for Applications (II)

What Are Multivariate Splines?

Let $\Delta$ be a triangulation of a domain $\Omega \subset \mathbb{R}^2$. For integers $d \geq 1$, $-1 \leq r \leq d$ define by

$$S_d^r(\Delta) = \{s \in C^r(\Omega), s|_t \in \mathbf{P}_d, t \in \Delta\}$$

the spline space of smoothness $r$ and degree $d$ over $\Delta$.

In general, let $r = (r_1, \cdots, r_n)$ with $r_i \geq 0$ be a vector of integers. Define

$$S_d^r(\Delta) = \{s \in C^{-1}(\Omega), s|_{e_i} \in C^{r_i}, e_i \in E\},$$

where $E$ is the collection of interior edges of $\Delta$. Each spline in $S_d^r(\Delta)$ has variable smoothness.

This can handle the situation of hanging nodes in a triangulation!
Definition of Spline Functions

Let $T = \langle (x_1, y_1), (x_2, y_2), (x_3, y_3) \rangle$. For any point $(x, y)$, let $b_1, b_2, b_3$ be the solution of

\[
\begin{align*}
x &= b_1 x_1 + b_2 x_2 + b_3 x_3 \\
y &= b_1 y_1 + b_2 y_2 + b_3 y_3 \\
1 &= b_1 + b_2 + b_3.
\end{align*}
\]

Fix a degree $d > 0$. For $i + j + k = d$, let

\[
B_{ijk}(x, y) = \frac{d!}{i!j!k!} b_1^i b_2^j b_3^k
\]

which is called Bernstein-Bézier polynomials. For each $T \in \Delta$, let

\[
S|_T = \sum_{i+j+k=d} c_{ijk}^T B_{ijk}(x, y).
\]

We use $s = (c_{ijk}^T, i + j + k = d, T \in \Delta)$ be the coefficient vector to denote a spline function in $S_d^{-1}(\Delta)$. 
We use the de Casteljau algorithm to evaluate a Bernstein-Bézier polynomial at any point inside the triangle. It is a simple and stable computation.

Let $T = \langle v_1, v_2, v_3 \rangle$ and $S|_T = \sum_{i+j+k=d} c_{ijk} B_{ijk}(x, y)$. Then directional derivative

$$D_{v_2-v_1} S|_T = d \sum_{i+j+k=d-1} (c_{i,j+1,k} - c_{i+1,j,k}) B_{ijk}(x, y).$$

Similar for $D_{v_3-v_1} S|_T$.

$D_x$ and $D_y$ are linearly combinations of these two directional derivatives.
Let $T_1$ and $T_2$ be two triangles in $\Delta$ which share a common edge $e$. Then $S \in C^r(T_1 \cup T_2)$ if and only if the coefficients of $c^{T_1}_{ijk}$ and $c^{T_2}_{ijk}$ satisfy the following linear conditions. E.g.,

$$S \in C^0(T_1 \cup T_2) \text{ iff } c^{T_1}_{0,j,k} = c^{T_2}_{j,k,0}, j + k = d$$

$$S \in C^1(T_1 \cup T_2) \text{ iff } c^{T_1}_{1,j,k} = b_1 c^{T_2}_{j+1,k,0} + b_2 c^{T_2}_{j,k+1,0} + b_3 c^{T_2}_{j,k,1}$$

for $i + k = d - 1$ and etc. (cf. [Farin, 86] and [de Boor, 87]). We code them by $Hc=0$. 
Integration

Let $s$ be a spline in $S_d^r(\triangle)$ with

$$s|_T = \sum_{i+j+k=d} c_{ijk}^T B_{ijk}(x, y), T \in \triangle.$$  

Then

$$\int_{\Omega} s(x, y) \, dxdy = \sum_{T \in \triangle} \frac{A_T}{(d+2)/2} \sum_{i+j+k=d} c_{ijk}^T.$$  

If $p = \sum_{i+j+k=d} a_{ijk} B_{ijk}(x, y)$ and $q = \sum_{i+j+k=d} b_{ijk} B_{ijk}(x, y)$ over a triangle $T$, then

$$\int_T p(x, y)q(x, y) \, dxdy = a^\top M_d b,$$

where $a = (a_{ijk}, i+j+k = d)^\top$, $b = (b_{ijk}, i+j+k = d)^\top$, $M_d$ is a symmetric matrix with known entries (cf. [Chui and Lai, 1992]). Similarly, we have (cf. [Awanou and Lai, 2005])

$$\int_T p(x, y)q(x, y)r(x, y) \, dxdy = a^\top A_d b \odot c.$$
Spline Approximation Order

We have (cf. [Lai and Schumaker’98]\(^6\))

**Theorem**

*Suppose that \( \triangle \) is a \( \beta \)-quasi-uniform triangulation of domain \( \Omega \in \mathbb{R}^2 \) and suppose that \( d \geq 3r + 2 \). Fix \( 0 \leq m \leq d \). Then for any \( f \) in a Sobolev space \( W_p^{m+1}(\Omega) \), there exists a quasi-interpolatory spline \( Q_f \in S_d^r(\triangle) \) such that*

\[
\| f - Q_f \|_{k,p,\Omega} \leq C |\triangle|^{m+1-k} |f|_{d+1,p,\Omega}, \quad \forall 0 \leq k \leq m + 1,
\]

*for a constant \( C > 0 \) independent of \( f \), but dependent on \( \beta \) and \( d \).*

See more detail in monograph\(^7\)

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The Weak Form of PDE’s

The weak formulation for elliptic PDE’s reads: find \( u \in H^k(\Omega) \) which satisfies its boundary condition such that

\[
a(u,v) = \langle f,v \rangle, \quad \forall v \in H^r_0(\Omega), \tag{1}
\]

where \( a(u,v) \) is the bilinear form defined by

\[
a(u,v) = \begin{cases} 
\int_{\Omega} \nabla u \cdot \nabla v \, dx \, dy, & k = 1; \\
\int_{\Omega} \Delta u \Delta v \, dx \, dy, & k = 2,
\end{cases}
\]

and \( \langle f,v \rangle = \int_{\Omega} f(x,y)v(x,y) \, dx \, dy \) is the \( L_2 \) inner product of \( f \) and \( v \). Here \( H^k(\Omega) \) and \( H^r_0(\Omega) \) are standard Sobolev spaces.

Clearly, our model problem is the Euler-Lagrange equation associated with a minimization problem:

\[
\min_{u \in H^k(\Omega)} E(u), \ u \text{ satisfies given boundary conditions} \tag{2}
\]

where the energy functional \( E(u) = \frac{1}{2} a(u,u) - \langle f,u \rangle \).
**Our Spline Method**

[1] We write $s \in S_d^{-1}(\triangle)$ in

$$s(x, y)|_t = \sum_{i+j+k=d_t} c_{i,j,k}^t B_{i,j,k}^t(x, y), \quad (x, y) \in t \in \triangle.$$ 

Let $c = (c_{i,j,k}^t, i + j + k = d_t, t \in \triangle)$ be the B-coefficient vector associated with $s$.

[2] We compute mass and stiffness matrices:

$$E(s) = \frac{1}{2} c^\top K c - c^\top M f$$

where $K$ and $M$ are diagonally block matrices.

[3] Since $s \in S_d^r(\triangle)$, we have the smoothness conditions $H c = 0$. Also, the boundary condition can be written as $B c = g$.

[4] Finally we solve the constrained minimization problem:

$$\min E(s), \text{ subject to } H c = 0, \ B c = g.$$
By using Lagrange multiplier method, we let

\[ L(c, \alpha, \beta) = \frac{1}{2} c^T K c - c^T M f + \alpha H c + \beta (B c - g) \]

and compute local minimizers. That is, we solve

\[
\begin{bmatrix}
B^T & H^T & K \\
0 & 0 & B \\
0 & 0 & H
\end{bmatrix}
\begin{bmatrix}
\beta \\
\alpha \\
c
\end{bmatrix} =
\begin{bmatrix}
M f \\
g \\
0
\end{bmatrix}
\]

which can be solved using the least squares method or by an iterative algorithm in [Awanou, Lai, Wenston’06\(^8\)].

We minimize

\[ \tilde{L}(c, \alpha, \beta) = \frac{1}{2} c^T K c - c^T M f + \alpha \| H c \|^2 + \beta \| B c - g \|^2. \]

An Iterative Algorithm

We begin with

\[
\left( K + \frac{1}{\epsilon}[B^T, H^T]^T[B^T H^T] \right) c^{(1)} = Mf + \frac{1}{\epsilon}[B^T, H^T]^T G
\]

Then we do the following iterative step

\[
\left( K + \frac{1}{\epsilon}[B^T, H^T]^T[B^T H^T] \right) c^{(k+1)} = Kc^{(k)} + \frac{1}{\epsilon}[B^T, H^T]^T G
\]

for \( k = 1, 2, \cdots, 10 \) and \( \epsilon > 0 \), e.g., \( \epsilon = 10^{-6} \) with \( c = 0 \).

Theorem (Awanou, Lai, and Wenston’06)

Suppose that \( K \) is positive definite with respect to \([B^T H^T]\). Then the above iterative algorithm converges and

\[
\|c^{(k+1)} - c\| \leq C\epsilon^k, \quad \forall k \geq 1.
\]
A natural question follows: can we define splines on a partition containing not only triangles, but also other polygons, say over a Voronoi diagram (i.e. Dirichlet tessellation) or a patio?

That is, can we be more versatile in solution of PDE? Can we be more efficient than the standard finite element method?

These questions have been raised in the community of numerical solution of PDE. The so-called virtual element method and the discontinuous Galerkin method are developed recently.
Generalized Barycentric Coordinates

Similar to the barycentric coordinates $b_1, b_2, b_3$ associated with a $T = \langle v_1, v_2, v_3 \rangle$, on an arbitrary polygon $P = \langle v_1, \cdots, v_n \rangle$, say pentagon

there are functions $b_1 \geq 0, \cdots, b_n \geq 0$

satisfying

$$\sum_{i=1}^{n} b_i(x) = 1 \text{ and } x = \sum_{i=1}^{n} b_i(x)v_i \quad \forall x \in \mathbb{R}^2$$

which are called generalized barycentric coordinates (GBC).
There are many kinds of GBC available. Construction of various GBC is pioneered by Wachspress in 1975. More and more GBC have been invented.

- Wachspress Coordinates,
- Mean Value Coordinates,
- Bilinear Coordinates,
- maximum entropy coordinates,
- discrete harmonic coordinates,
- Wachspress and mean value coordinates have 3D generalization and been extended in the spherical setting.

We refer to [Floater, 2015⁹] and [G. Manzini, A. Russo and N. Sukumar 2014¹⁰] for a summary surveying various types of GBC.

Wachspress Coordinates

Example (Wachspress coordinates)

For any polygon $P_n$ with $n$ sides, let

$$A_i(x) = A(x, v_i, v_{i+1}), \quad C_i = A(v_{i-1}, v_i, v_{i+1})$$

be the signed area of triangles $\langle x, v_i, v_{i+1} \rangle$ and $\langle v_{i-1}, v_i, v_{i+1} \rangle$, respectively. Setting $w_i(x) = C_i / (A_{i-1}(x)A_i(x))$ we define

$$\phi_i(x) = \frac{w_i(x)}{\sum_{j=1}^{n} w_j(x)}, \quad i = 1, 2, \ldots, n \quad (5)$$

which are called Wachspress barycentric coordinates. One can check these $\phi_i$ satisfy (4).

These coordinates have been developed in [Wachspress, 1975], [Warren, 1996], ... . The above form was given in [Meyer et al., 2002\textsuperscript{11}]

Polyangular Bernstein-Bézier Functions

For each $n$-gon $P_n$ and for any GBC $b_1, \cdots, b_n$, we let

$$B^d_j(x) = \frac{d!}{j!} \prod_{i=1}^{n} b^j_i, \quad |j| = d$$  \hspace{1cm} (6)

be the polyanglar Bernstein-Bézier Functions, where $j = (j_1, \cdots, j_n), j_i, i = 1, \cdots, n$ are nonnegative integers, $j! = \prod_{i=1}^{n} j_i!$ and $|j| = j_1 + \cdots + j_n$.

Let $\Phi_{n,d}$ be the linear space of functions of the form

$$s(x) = \sum_{|j|=d} c_j B^d_j(x), \quad x \in P_n$$  \hspace{1cm} (7)

with real coefficients $c_j$ and let

$$S^r_d(\Delta) = \{ s \in C^r(\Omega) : s|_{P_n} \in \Phi_{n,d}, P_n \in \Delta \}.$$  \hspace{1cm} (8)

be the polyanglar spline space of smoothness $r$ and degree $d$, where $\Delta$ is a collection of polygons.
Reduced Polygonal BB Functions

For convenience, we restrict ourselves to those BB functions in $\Phi_{n,d}(P)$ which enable us to reproduce polynomials of degree $d$. Let $\Psi_{d}(P_n)$ be the collection of all those terms in $\Phi_{d}(P_n)$ such that they are able to reproduce all polynomials of degree $d$. Any such functions in $\Psi_{d}(P_n)$ are called reduced Bernstein-Bézier functions.

Thus, let us identify these functions in $\Psi_{d}(P_n)$.

Recall that any $p \in \Pi_d$ has a unique, $d$-variate blossom, $\mathbb{P}[p]$. For $p$, its blossom $\mathbb{P}[p](x_1, \ldots, x_d)$ with variables $x_1, \ldots, x_d \in \mathbb{R}^2$ is uniquely defined by the three properties: (i) it is symmetric in the variables $x_1, \ldots, x_d$; (ii) it is multi-affine, i.e., affine in each variable while the others are fixed; and (iii) it has the diagonal property,

$$\mathbb{P}[p](x, \ldots, x) = p(x).$$
Example 1

Consider \( d = 2 \) first. Let us first expand the \( x \) variable using the barycentric coordinates of \( x \) with respect to the vertex \( v_i \) and its two neighbors, i.e., with respect to \( T_i := [v_{i-1}, v_i, v_{i+1}] \). That is,

\[
x = \sum_{j=-1}^{1} \lambda_{i,j}(x)v_{i+j}, \quad 1 = \sum_{j=-1}^{1} \lambda_{i,j}(x).
\]  

(9)

By the properties (4) and (9), we have

\[
p(x) = \mathbb{P}[p](x, x) = \sum_{i=1}^{n} b_i(x)\mathbb{P}[p](v_i, x) = \sum_{i=1}^{n} b_i(x) \sum_{j=-1}^{1} \lambda_{i,j}(x)\mathbb{P}[p](v_i, v_{i+j}).
\]
Example 1 (Cont.)

After a simplification, we have

\[ p(x) = \sum_{i=1}^{n} b_i(x) \lambda_{i,0}(x) p(v_i) + \]

\[ \sum_{i=1}^{n} \mathbb{P}[p](v_i, v_{i+j})(b_i(x) \lambda_{i,1}(x) + b_{i+1}(x) \lambda_{i+1,-1}(x)). \]  \hfill (10)

Thus, we are ready to define

\[ F_i(x) = b_i(x) \lambda_{i,0}(x) \] and \[ L_i = b_i(x) \lambda_{i,1}(x) + b_{i+1}(x) \lambda_{i+1,-1}(x). \]

\hfill (11)

We know that span\{\( F_i, L_i, i = 1, \cdots, n \)\} is able to reproduce \( \Pi_2 \).
Also, it is clear that each \( \lambda_{i,j} \) can be expressed by using \( b_k, k = 1, \cdots, n \) and hence, \( F_i, L_i \in \Phi_2(P_n) \). These are reduced BB functions.
Reduced BB Functions of Order $d \geq 3$

Let

$$F_i = \lambda_{i,0}^{d-1}, \quad F_{i,k} = \binom{d-1}{k} \phi_i \lambda_{i,1}^k \lambda_{i,0}^{d-1-k} + \binom{d-1}{k-1} \phi_{i+1} \lambda_{i,-1}^{d-k} \lambda_{i,1}^{k-1}.$$ 

**Theorem (Floater and Lai, 2015$^{12}$)**

For $d \geq 3$ and for Wachspress coordinates, the reduced BB function space $\Psi_d(P_n)$ is

$$\text{span}\{F_i, i = 1, \ldots, n\} \oplus \text{span}\{F_{i,k}, i = 1, \ldots, n, k = 1, \ldots, d-1\} \oplus \frac{b}{W} \prod_{d-3},$$

where $P_n$ and $b(x) = \prod_{j=1}^{n} h_j(x)$.

---

Let $\Delta$ be a collection of polygons with arbitrary number sides. Let $
abla = \bigcup_{P \in \Delta} P$ be the domain consisting of these polygons in $\Delta$. We assume that the interiors of any two polygons $P$ and $\tilde{P}$ from $\Delta$ do not intersect and the intersection of $P$ and $\tilde{P}$ is either their common edge $e$ or common vertex $v$ if the intersection is not empty. We can define a continuous polygonal spline space of order $d \geq 1$ over $\Omega$ by

$$S_d(\Delta) = \{ s \in C(\Omega), s|_{P_n} \in \Psi_d(P_n), \forall P_n \in \Delta \},$$

where for $d = 1$, we simply let $\Psi_1(P_n) = \text{span}\{ \phi_i, i = 1, \cdots, n \}$ and for $d \geq 2$, we use the reduced BB function space $\Psi_d(P_n)$ discussed above.
We can establish the dimension of $S_d(\Delta)$.

**Theorem**

Suppose that $S_d(\Delta)$ is a polygonal spline space based on Wachspress coordinates. Then the dimension of $S_d(\Delta)$ is

$$\dim(S_d(\Delta)) = #(V) + (d - 1)#(E) + #(\triangle)(d - 1)(d - 2)/2,$$

where $V$ is the collection of all vertices of $\Delta$, $E$ is the collection of all edges of $\Delta$ and $(\triangle)$ stands for the number of polygons in $\Delta$.

**Open Question**: How to construct $C^r$ smooth spline functions over a polygonal mesh $\Delta$ for $r \geq 1$?
An Extension to the Trivariate Setting

For simplicity, let us consider a hexahedron $H \subset \mathbb{R}^3$. It is a polyhedron with 6 faces, 12 edges and 8 vertices.

Advantages:
(1) $H$ is a simple polyhedron, three incidental faces at each vertex of $H$.
(2) One can use such hexahedrons to partition any polyhedral domain in $\mathbb{R}^3$. 
Hexahedral Partitions

Theorem

We can use hexahedrons to partition any polyhedral domain $\Omega$ in $\mathbb{R}^3$.

Proof.

First for any tetrahedron $T$, one can decompose it into 4 hexahedrons.

As one can use tetrahedrons to partition any polyhedral domain $\Omega \in \mathbb{R}^3$, it follows that we can use hexahedrons to partition $\Omega$. □
Let $\Delta$ be a collection of hexahedrons which partitions a polyhedral domain $\Omega \in \mathbb{R}^3$. Let

$$S_d(\Delta) = \{ s \in C(\Omega), s|_H \in \Psi_d(P_n), \forall H \in \Delta \}, \quad (14)$$

where $\Psi_d(H)$ is the reduced BB function space for $d \geq 1$. With a straightforward extension, we are able to prove the following

**Theorem**

*For any fixed generalized barycentric coordinates, let $\Psi_2(H)$ be the space spanned by the reduced Bernstein-Bézier functions. Then the dimension of $S_2(\Delta)$ is*

$$\dim(S_2(\Delta)) = #(V) + #(E),$$

*where $V$ is the collection of all vertices of $\Delta$ and $E$ is the collection of all edges of $\Delta$.***
We now present an application of these polygonal splines for numerical solution of Poisson equation with Dirichlet boundary value problem:

\[
\begin{cases}
-\Delta u = f, & x \in \Omega \subset \mathbb{R}^2 \\
u = g, & x \in \partial\Omega,
\end{cases}
\]  

\hspace{1cm} (15)

where \( \Omega \) is a polygonal domain and \( \Delta \) is the standard Laplace operator. Let \( S_2(\Delta) \) be the collection of polygonal spline functions of degree 2 over \( \Delta \). Let \( B(u, v) = \int_\Omega \nabla u \cdot \nabla v \, dx \) be the standard bilinear form associated with Poisson equation. Our numerical method is to solve the following weak solution:

\[
B(u_h, v) = \langle f, v \rangle, \forall v \in S_2(\Delta) \cap H^1_0(\Omega). 
\]  

\hspace{1cm} (16)
Numerical Solutions over Quadrilaterals

We have done several tests on smooth solutions:
\[ u_1(x, y) = \frac{1}{1 + x + y}, \quad u_2(x, y) = x^4 + y^4 \] and
\[ u_3(x, y) = \sin(\pi(x^2 + y^2)) + 1 \] to check the performance of our algorithm.

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Table: Convergence of serendipity quadratic finite elements over uniformly refined quadrangulations

The convergence rate is 3, consistent with the theory and the numerical results from [Rand, Gillette and Babaj, 2014].
Uniform Refinement of Pentagonal Partitions

Figure: An illustration of uniform pentagon refinement with vertices $v_i$, $u_i$, $w_i$, $i = 1, \ldots, 5$. The pentagonal partitions are uniformly refined, and the refinement process is depicted with additional vertices and edges.
Partition of Domains

**Figure**: A Pentagon Partition (top-left) and its refinements
Numerical Solution of PDE over Pentagonal Partitions

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Table: Convergence of $C^0$ quadratic serendipity finite elements over uniformly refined pentagon partitions in Fig. 2.
Ming-Jun Lai

Motivation
Polynomial Splines
Spline Method for PDE
GBC
BB functions
Polygonal Spline Space
Hexahedral Spline Space
Solution of PDE

A Numerical Solution over Mixed Polygons
Hexahedral Spline Approximation

Recall the definition of $F_i$ associated with vertex $v_i$ of a hexahedron $H$ and $L_i$ associated with edge $e_i$ of $H$. For a collection $\Delta$ of hexahedraons, we can define a locally supported polyhedral spline function $F_v$ associated with each vertex $v$ of $\Delta$ and $L_e$ associated with each edge.
Hexahedral Spline Approximation (II)

We define a quasi-interpolatory operator

$$Q(f) = \sum_{v \in \Delta} f(v)F_v + \sum_{e \in \Delta} f(u_e)L_e$$

where $u_e$ is the midpoint of the edge $e$. Then we have

**Theorem**

*For all quadratic polynomial $p$ in total degree,*

$$Q(p) = p.$$
Conclusion and Future Research

There are a lot of research remaining open:

- dimensions of polygonal BB functions, dimension of polygonal spline space for other GBC’s $d \geq 3$ than Wachspress coordinates.
- how to define $C^r$ smooth spline spaces with $r \geq 1$, what are the approximation power of those spline spaces.
- Why is the approximation rate based on pentagonal refinement scheme faster?
- Refinability: let $\Delta_2$ be a uniform refinement of $\Delta$. Is $S_d(\Delta) \subset S_d(\Delta_2)$?
- We are extending our study to the 3D setting, in particular, numerical solution of PDE.
Thanks for your attention!