NUMERICAL INTEGRATION of HOMOGENEOUS FUNCTIONS and POLYNOMIALS on POLYTOPES

Eric B. Chin
UC DAVIS

Jean B. Lasserre
LAAS-CNRS

N. Sukumar
UC DAVIS

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Problem statement

Determine

\[ \int_{P \subset \mathbb{R}^d} f(x) \, dx \]

- \( f(x) \) is a homogeneous function
- \( P \) is a convex or nonconvex polytope
Existing methods

Three methods to integrate functions on polytopes

- Triangulation
- Divergence theorem
  \[
  \int_V \nabla \cdot F \, dV = \int_S F \cdot n \, dS
  \]
- Moment fitting
Background

Euler’s homogeneous function theorem and Generalized Stokes’s theorem
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Extension of Lasserre’s method

Integration over convex and nonconvex polytopes
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Conclusions and outlook
Background
A continuously differentiable function $f(x)$ is said to be positive homogeneous of degree $q$ if:

$$f(\lambda x) = \lambda^q f(x) \quad \forall \lambda > 0,$$

and then it also satisfies:

$$q f(x) = \langle \nabla f(x), x \rangle \quad \forall x \in \begin{cases} \mathbb{R}^d & \text{if } q > 0 \\ \mathbb{R}^d \setminus \{0\} & \text{if } q < 0 \end{cases}$$

$\langle \cdot, \cdot \rangle$: inner product \quad $d$: dimension
Proof

By definition, a homogeneous function of degree \( q \) has the property

\[
\lambda^q f(x) = f(\lambda x)
\]

Define \( x' := \lambda x \) and calculate \( \frac{\partial}{\partial \lambda} \):

\[
q \lambda^{q-1} f(x) = \frac{\partial f}{\partial x'} \cdot \frac{\partial x'}{\partial \lambda} = \frac{\partial f}{\partial x'} \cdot x
\]

Let \( \lambda = 1 \):

\[
q f(x) = \frac{\partial f}{\partial x} \cdot x = \langle \nabla f(x), x \rangle
\]

Converse is also readily established
Examples of homogeneous fns.

\[ q = 0: \]
\[ f(x) = 1 \]

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\[ f(x) = x + y \]

\[ q = 2: \]
\[ f(x) = 3x^2 + 2xy \]
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\[ q = 2: \]
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\[ q = -\frac{1}{2}: \]
\[ f(x) = \frac{1}{\sqrt{r}}, \quad \text{where} \]
\[ r = \sqrt{x^2 + y^2} \]

\[ q = 0: \]
\[ f(x) = \cos \theta, \quad \text{where} \]
\[ \theta = \tan^{-1} \frac{y}{x} \]

\[ q = \frac{1}{2}: \]
\[ f(x) = \sqrt{r} \cos \theta \]
Generalized Stokes’s theorem

\[
\int_M d\omega = \int_{\partial M} \omega
\]

\[
\downarrow
\]

\[
\int_M (\nabla \cdot X) f(x) \, dx + \int_M X \cdot \nabla f(x) \, dx = \int_{\partial M} (X \cdot n) f(x) \, d\sigma
\]

- \(X\): vector field
- \(M\): region of integration
- \(d\sigma\): Lebesgue measure on \(\partial M\)
Extension of Lasserre’s method

First applied to X-FEM by Mousavi and S (Comp. Mech., 2011)

Extended to nonconvex regions by Chin et al. (Comp. Mech., 2015, doi 10.1007/s00466-015-1213-7) [PDF]

Method uses properties of homogeneous functions and generalized Stokes’s theorem
Main results
Reducing integration to bdry.

Apply Stokes’s theorem with $X := x$ and $f (x)$ is $q$-homogeneous:

$$d \int_P f (x) \, dx + \int_P \langle \nabla f (x) , x \rangle \, dx = \sum_{i=1}^{m} \int_{F_i} (x \cdot n_i) \, f (x) \, d\sigma$$

$P$: polygon \hspace{1cm} $F_i$: boundary facets

Apply Euler’s homogeneous fn. theorem, $qf (x) = \langle \nabla f (x) , x \rangle$:

$$d \int_P f (x) \, dx + q \int_P f (x) \, dx = \sum_{i=1}^{m} \int_{F_i} (x \cdot n_i) \, f (x) \, d\sigma$$

$$\int_P f (x) \, dx = \frac{1}{d+q} \sum_{i=1}^{m} \int_{F_i} (x \cdot n_i) \, f (x) \, d\sigma$$
\[ \int_{P} f(\mathbf{x}) \, d\mathbf{x} = \frac{1}{d + q} \sum_{i=1}^{m} \int_{F_{i}} (\mathbf{x} \cdot \mathbf{n}_{i}) f(\mathbf{x}) \, d\sigma \]

- \( F_{i} \subset \mathbf{a}_{i} \cdot \mathbf{x} = b_{i} \): equation of a hyperplane
- \( \mathbf{n}_{i} = \frac{\mathbf{a}_{i}}{\|\mathbf{a}_{i}\|} \): unit normal to hyperplane
- \( \mathbf{x} \cdot \mathbf{n}_{i} = \mathbf{x} \cdot \frac{\mathbf{a}_{i}}{\|\mathbf{a}_{i}\|} = \|\mathbf{a}_{i}\| = \frac{b_{i}}{\|\mathbf{a}_{i}\|} \)

\[ \therefore \quad \int_{P} f(\mathbf{x}) \, d\mathbf{x} = \frac{1}{d + q} \sum_{i=1}^{m} \int_{F_{i}} \frac{b_{i}}{\|\mathbf{a}_{i}\|} f(\mathbf{x}) \, d\sigma \quad (*) \]

- Using \((*)\), one can reduce integration to the boundary of the polytope
**Sign of** \(a_i\) **and** \(b_i\)

**Question:** \(x = 1\) and \(-x = -1\) produce the same line, yet only one gives the correct answer in \((*)\). Which is correct?

**Answer:** Given the vertices of the polygon, travel around the polygon in a **clockwise** direction.
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**Answer:** Given the vertices of the polygon, travel around the polygon in a *clockwise* direction.
Sign of $a_i$ and $b_i$

$$\det \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = (y_1 - y_2)x + (x_2 - x_1)y + (x_1y_2 - y_1x_2)$$

$$a_i = \{y_1 - y_2, x_2 - x_1\}^T \quad b_i = -(x_1y_2 - y_1x_2)$$
Further reducing the integration

Reapplying Stokes’s theorem

\[ \int_M (\nabla \cdot X) f(x) \, dx + \int_M X \cdot \nabla f(x) \, dx = \int_{\partial M} (X \cdot n) f(x) \, d\sigma \]

Select:

- \( M \) as \( F_i \)
- \( \partial M \) as \( v_{ij} \) (vertices in \( \mathbb{R}^2 \) – intersection of \( F_i \) and \( F_j \))
- \( X := x - x_0 \) (\( x_0 \): any point on hyperplane containing \( F_i \))
- \( f(x) \) is a homogeneous function of degree \( q \)
Integral of $f(x)$ on $F_i$

Reapplying Stokes’s theorem

\[
\int_M (\nabla \cdot X) f(x) \, dx + \int_M X \cdot \nabla f(x) \, dx = \int_{\partial M} (X \cdot n) f(x) \, d\sigma
\]

\[
\Rightarrow
\]

\[
\int_{F_i} f(x) \, d\sigma = \frac{1}{d + q - 1} \left[ \sum_{j=1}^{2} d_{ij} f(v_{ij}) + \int_{F_i} \langle \nabla f(x), x_0 \rangle \, d\sigma \right]
\]

(\*\*)

- $d_{ij} := \langle x - x_0, n_{ij} \rangle$ – algebraic distance from $v_{ij}$ to $x_0$
- $\int_{F_i} \langle \nabla f(x), x_0 \rangle \, d\sigma$ can be applied recursively
\[
\int_{\mathcal{P}} f(\mathbf{x}) \, d\mathbf{x} = \frac{1}{d + q} \sum_{i=1}^{m} \int_{F_i} \frac{b_i}{\|a_i\|} f(\mathbf{x}) \, d\sigma 
\]

\[
\int_{F_i} f(\mathbf{x}) \, d\sigma = \frac{1}{d + q - 1} \left[ \sum_{j=1}^{2} d_{ij} f(\mathbf{v}_{ij}) + \int_{F_i} \langle \nabla f(\mathbf{x}), \mathbf{x}_0 \rangle \, d\sigma \right]
\]

- These formulas can be used to reduce integration to the vertices of the polytope
- Further, a closed-form cubature rule can be developed
- If the partial derivatives of \( f(\mathbf{x}) \) eventually vanish, this cubature rule is exact
Combine (*) with (**):

\[
\int_P f(\mathbf{x}) \, d\mathbf{x} = \sum_{i=1}^{m} \frac{b_i}{\|a_i\|} \sum_{j \neq i} d_{ij} I(\mathbf{v}_{ij}) \frac{1}{(q + 2)(q + 1)}
\]

where

\[
I(\mathbf{v}_{ij}) := \sum_{k=0}^{q} \frac{Q_k(\mathbf{v}_{ij})}{(q)} \quad \text{and} \quad Q_k(\mathbf{v}_{ij}) := \sum_{|\alpha|=q-k} \frac{D^{\alpha}}{\alpha!} f(\mathbf{v}_{ij}) \prod_{\ell=1}^{2} (x_{0\ell})^{\alpha_{\ell}}
\]

- \(\alpha\) is an \(n\)-tuple of nonnegative integers
- \(D\) is the differential operator in multi-index notation
Consider a region $V$ bounded by homogeneous functions $h_i(x) = b_i$ ($i = 1, \ldots, m$) of degree $q_i$.

Apply Stokes’s theorem:

$$\int_V f(x) \, dx = \frac{1}{d + q} \sum_{i=1}^{m} q_i b_i \int_{A_i} \| \nabla h_i \|^{-1} f(x) \, d\sigma$$

Polar transformation:

$$\int_V f(x) \, dx = \frac{1}{2 + q} \sum_{i=1}^{m} \int_{\alpha_i}^{\beta_i} H_i^2(\theta) f(x(\theta)) \, d\theta$$

- Region bounded by equations of the form $r = H_i(\theta)$
Numerical examples
Integrate $f(x) = x^2 + xy + y^2$ over convex and nonconvex polygons

\[ \int_P f(x) \, dx \approx 323.1821 \quad \int_P f(x) \, dx \approx 80.95348 \]

Results verified in LattE (De Loera et al., Comput Geom, 2013)
Integration over polyhedra

Integrate $f(x) = x^2 + y^2 + z^2$ over nonconvex polyhedra

Octahedron 5-compound

\[ \int_{P} f(x) \, dx \approx 0.353553 \]

Echidnahedron

\[ \int_{P} f(x) \, dx \approx 253.5696 \]

Cube 5-compound

\[ \int_{P} f(x) \, dx \approx 1.250000 \]

▶ Shape data from PolyhedronData[] in Mathematica
Integration over curved region

\[ A := \{ r \in [0, 1], \theta \in [\pi/4, \pi/2], \]
\[ r \geq \cos \theta, r \leq \sin \theta, \theta \leq \pi/2 \} \]

\[ \int_A \frac{1}{\sqrt{x^2 + y^2}} = \sqrt{2} - 1 \]

- Weakly singular integrand at the origin
- Using equation derived for a curved region, domain integral is transformed to 1D line integrals
- With 6 quadrature points, integration error is close to machine precision
Application: X-FEM (Chin & S, work in progress, 2015)
Motivation

Numerical integration in elements with discontinuous and weakly singular integrands

Current approach

New approach: without partitioning!
Discontinuous + singular ints.

- Use (*) to reduce integration, then apply quadrature
- \( f(\mathbf{x}) = \sin(\theta/2)/\sqrt{r} \): discontinuous and weakly singular
- Biunit square centered at \((0.5, 0.5)\)

Application: X-FEM (Chin & S, work in progress, 2015)
Elastic Fracture

**Strong form**

\[ \nabla \cdot \sigma = 0 \text{ in } \Omega \]
\[ u = \bar{u} \text{ on } \Gamma_u \]
\[ \sigma \cdot n = \bar{t} \text{ on } \Gamma_t \]
\[ \sigma \cdot n = 0 \text{ on } \Gamma_c \]

**Weak form**

\[ a(u, \delta u) = \ell(\delta u) \quad \forall \delta u \in \mathcal{U}_0, \]

\[ a(u, \delta u) := \int_{\Omega} \sigma : \delta \varepsilon \, dx, \quad \ell(\delta u) := \int_{\Gamma_t} \bar{t} \cdot \delta u \, dS \]
Displacement approximation

Standard FEM

\[ u(x) = \sum_{i \in \mathcal{I}} N_i(x) u_i \]

Extended FEM [X-FEM] (Moës et al., IJNME, 1999)

\[ u(x) = \sum_{i \in \mathcal{I}} N_i(x) u_i + \sum_{j \in \mathcal{J} \subseteq \mathcal{I}} N_j(x) \varphi(x) a_j + \sum_{k \in \mathcal{K} \subseteq \mathcal{I}} N_k(x) \sum_{t=1}^{2} \sum_{\alpha=1}^{4} F_{\alpha t}(x) b_{k\alpha t} \]

- \( u_i, a_j, b_{k\alpha t} \) - degrees of freedom (DOFs)
- \( \varphi(x) \) - discont. enrichment (usually \( \varphi(x) := H(x) = \begin{cases} 1 & x \in \Omega_e^+ \\ -1 & x \in \Omega_e^- \end{cases} \))
- \( F_{\alpha t}(x) \) - crack-tip enrichment

Application: X-FEM (Chin & S, work in progress, 2015)
**Application: Extraction of SIFs**

**Mode I SIF for an inclined, embedded crack**

Integrand in the interaction integral are homogeneous functions ⇒ can use Lassere’s approach to compute SIFs

For the results shown above, Triangulation is used to compute the interaction integral in cracked elements

<table>
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<th>$\beta$</th>
<th>$K_1$ (exact)</th>
<th>$K_1$ (GD)</th>
<th>$K_1$ (CLS)</th>
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<td>90°</td>
<td>2.5066</td>
<td>2.5095</td>
<td>2.5088</td>
</tr>
</tbody>
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- Showcased an application in elastic fracture using the X-FEM
- New integration scheme is well-suited for adoption in novel discretization techniques such as MFD, VEM, WG, DG, PUFEM, embedded interface methods, . . .